**ASE 380P ANALYTICAL METHODS I**
**EM386K MATHEMATICAL METHODS IN APPLIED MECHANICS I**

**Exam 1. Monday, October 17, 2011**

1. (a) Formulate Leibnitz's formula for computing the derivative (w.r.t. $x$) of the integral

$$\int_{\alpha(x)}^{\beta(x)} f(x,t)\, dt$$

(3 points).

$$\frac{d}{dx}\left(\int_{\alpha(x)}^{\beta(x)} f(x,t)\, dt\right) = \int_{\alpha}^{\beta} \frac{\partial f}{\partial x}(x,t)\, dt + f(\beta(x),x)\beta' - f(\alpha(x),x)\alpha'$$

(b) Argue why the integral

$$\int_0^1 \ln^3 t\, dt$$

is finite (5 points).

This may be done in many ways. Here is one. De Hospital's rule implies that, for any $\epsilon > 0$,

$$\lim_{t\to 0} \frac{\ln t}{t^{-\epsilon}} \stackrel{H}{=} \lim_{t\to 0} \frac{t^{-1}}{-\epsilon t^{-\epsilon-1}} = -\frac{1}{\epsilon}\lim_{t\to 0} t^{\epsilon} = 0$$

Thus, $\frac{\ln t}{t^{-\epsilon}}$ is bounded for $t \in [0,1]$, say

$$|\frac{\ln t}{t^{-\epsilon}}| \le M$$

This implies,

$$|\int_0^1 \ln^3 t\, dt| \le M^3 \int_0^1 t^{-\frac{3}{\epsilon}}\, dt < \infty \quad \text{for } \frac{3}{\epsilon} < 1$$

(c) Evaluate the integral above. *Hint:* Differentiate the known result

$$\int_0^1 t^{\gamma}\, dt = (\gamma+1)^{-1}$$

repeatedly with respect to $\gamma$ (17 points).

Recall how to differentiate exponential function,

$$\frac{d}{d\gamma} t^{\gamma} = t^{\gamma} \ln t$$

Differentiating the result above three times and using the Leibnitz rule, we get,

$$\frac{\partial^3}{d\gamma^3}\int_0^1 t^\gamma\, dt = \int_0^1 t^\gamma \ln^3 t\, dt = -6(\gamma+1)^{-4}$$

Set $\gamma = 0$ to get

$$\int_0^1 \ln^3 t\, dt = -6$$

2. (a) Define the Cauchy Principal Value (CPV) integral for $\int_a^b f(x)\,dx$, where integrand $f(x)$ is singular at point $c \in (a,b)$, and for integral $\int_{-\infty}^{\infty} f(x)\,dx$ where the integrand need not be singular but the integration extends over the entire real line. (5 points).

$$\text{(CPV)}\int_a^b f(x)\,dx = \fint_a^b f(x)\,dx := \lim_{\epsilon\to 0}\left\{\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\epsilon}^b f(x)\,dx\right\}$$

and

$$\fint_{-\infty}^{\infty} f(x)\,dx := \lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$$

(b) Determine whether the following integral exists

$$\text{(CPV)}\int_0^\infty \frac{\cos x}{\ln x}\,dx$$

(20 points). Start by decomposing the guy into three integrals,

$$\fint_0^\infty \frac{\cos x}{\ln x}\,dx = \int_0^{1/2}\frac{\cos x}{\ln x}\,dx + \fint_{1/2}^{3/2}\frac{\cos x}{\ln x}\,dx + \int_{3/2}^{\infty}\frac{\cos x}{\ln x}\,dx$$

Now reason:

- The first integral exists in the usual sense, since the integrand is bounded.
- The last integral exists (in the usual sense) by the Dirichlet criterion. Indeed,

$$\frac{1}{\ln x} \searrow 0 \text{ as } x\to\infty$$

and $\cos x$ oscilates so $\int_{3/2}^{x} \cos s\, ds$ is uniformly bounded in $x$.
- It is only the second integral that has to be considered in the CPV sense. We need to converge somehow to Hölder continuity. Here is one way to get there:

$$\fint_{1/2}^{3/2}\frac{\cos x}{\ln x}\,dx = \fint_{1/2}^{3/2}\underbrace{\frac{\cos x(x-1)}{\ln x}}_{=:\phi(x)}\frac{1}{x-1}\,dx = \phi(1)\fint_{1/2}^{3/2}\frac{dx}{x-1} + \fint_{1/2}^{3/2}\frac{\phi(x)-\phi(1)}{x-1}\,dx$$

Now, the first integral is simply zero. Concerning the second,

i. $\phi(x)$ is differentiable in $[1/2, 3/2]$ and has bounded derivative. This intuitive statement is not entirely trivial and needs some effort to be shown. You can argue for instance like this:

   - $\cos x$ and its derivative: $-\sin x$ are bounded.
   - $\frac{x-1}{\ln x}$ and its derivative are also bounded. Here you have to compute the derivative and use the Hospital's rule to check that this is indeed true. This takes a couple of lines, fill the blanks...
   - Product of two differentiable guys with bounded derivatives is differentiable and its derivative is bounded, too.

ii. Now, any function with bounded derivative, is Lipschitz (Hölder with exponent 1). This is a consequnece of Mean-Value Theorem.

$$f(y) - f(x) = f'(\xi)(y - x), \quad \xi \in [x, y]$$

implies

$$|f(y) - f(x)| = \underbrace{|f'(\xi)|}_{\leq M} |y - x| \leq M|y - x|$$

The Lipschitz continuity is the key point in showing that the last integral exists in the classical sense,

$$|\int_{1/2}^{3/2} \frac{\phi(x) - \phi(1)}{x - 1}\, dx| \leq \int_{1/2}^{3/2} |\frac{\phi(x) - \phi(1)}{x - 1}|\, dx \leq M \int_{1/2}^{3/2} dx = M$$

3. (a) Define the notion of the adjoint operator (2 points).

Let $A : X \to Y$ be a linear operator where $X, (\cdot, \cdot)_X$, $Y, (\cdot, \cdot)_Y$ are Hilbert spaces. Operator $A^* : Y \to X$ is *adjoint* to operator $A$, iff

$$(Ax, y)_Y = (x, A^*y)_X, \quad \forall x \in X, y \in Y$$

(b) Compute the adjoint of operator

$$Lu = \frac{du}{dx}, \quad D(L) := \{u \in ? \ : \ 2u(0) = u(1)\}$$

with respect to $L^2$ inner product (real case),

$$(u, v) = \int_0^1 u(x)v(x)\, dx$$

(13 points) Is is just a matter of integration by parts,

$$\int_0^1 u'v\, dx = -\int_0^1 uv'\, dx + u(1)v(1) - u(0)v(0)$$

But

$$u(1)v(1) - u(0)v(0) = 2u(0)v(1) - u(0)v(0) = u(0)(2v(1) - v(0))$$

will vanish only if $2v(1) = v(0)$. Hence,

$$A^*v = -v', \quad D(A^*) = \{v \in ? \, : \, 2v(1) = v(0)\}$$

(c) Compute the adjoint of operator of the same operator but with respect to a different inner product,

$$(u, v) = \int_0^1 (1 + x^2)u(x)v(x) \, dx$$

(10 points)

Same exercise, but with different inner product,

$$\int (1+x^2)u'v = -\int u[(1+x^2)v]' + (1+x^2)uv|_0^1 = -\int (1+x^2)\frac{[(1+x^2)v]'}{1+x^2} + 2u(1)v(1) - u(0)v(0)$$

leads to

$$A^*v = -\frac{[(1+x^2)v]'}{1+x^2} = -v' - \frac{2x}{1+x^2}v, \quad D(A^*) = \{v \in ? \, : \, 4v(1) = v(0)\}$$

the moral of the story is that the adjoint does depend upon the inner product you are using.

4. (a) State the conditions that the right-hand side $\boldsymbol{y}$ has to satisfy (in terms of adjoints) so that the general linear problem

$$A\boldsymbol{x} = \boldsymbol{y}$$

has a solution (5 points).

Assume $x$ and $y$ come from Hilbert spaces. Multiply both sides of the equations with an arbitrary $z \in Y$ and use the definition of adjoint to conclude that

$$(\boldsymbol{x}, A^*\boldsymbol{z})_X = (A\boldsymbol{x}, \boldsymbol{z})_Y = (\boldsymbol{y}, \boldsymbol{z})_Y$$

If $A^*\boldsymbol{z} = 0$ then it must be then $(\boldsymbol{y}, \boldsymbol{z})_Y = 0$. In other words, the right-hand side must be orthogonal to null space of the adjoint operator. This necessary condition is also sufficient (most of the time...).

(b) Apply the theory to determine necessary and sufficient conditions for the case when $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ and $A$ represents the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & -4 \\ 1 & 1 & -6 \end{pmatrix}$$

Is the solution unique ? (20 points)

**Comment:** You may use first elementary means to determine the answer but eventually I want to see an argument based on the adjoints.

I can work with any inner product that I want, so the canonical inner product is the best, (the simplest) choice. Adjoint of a real-valued matrix wrt the canonical inner product is its transpose. So the first step is to determine the null space of the transpose, i.e. solve the homogenous system:

$$\boldsymbol{A}^*\boldsymbol{z} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & -4 & -6 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \mathbf{0}$$

The matrix is singular with rank 2, we get,

$$N(A^*) = \{(-t, t, -t) \ : \ t \in \mathbb{R}\} =: \mathbb{R}(-1, 1, -1)$$

The necessary and sufficient condition for the solution to exist is thus:

$$(-1, 1, -1) \cdot (y_1, y_2, y_3) = -y_1 + y_2 - y_3 = 0$$

The solution is not unique since $A$ is singular, i.e. the null space of $A$ is non-trivial, too. For a square matrix, nullity of $A$ = nullity of $A^*$ (=1 in this case). The solution can be determined thus up to one unknown scalar.