CES 389C/EM 397 INTRODUCTION TO MATHEMATICAL MODELING IN SCIENCE AND ENGINEERING
Final Exam, Dec 17, 2012

1. A "sanity check" for Lagrangians and Hamiltonians.

   (a) Consider a single particle constrained to move on a (smooth) parabola $x = y^2$ under its own weight (see Fig. 1). Use $q := y$ for a generalized coordinate, write down the formula for the

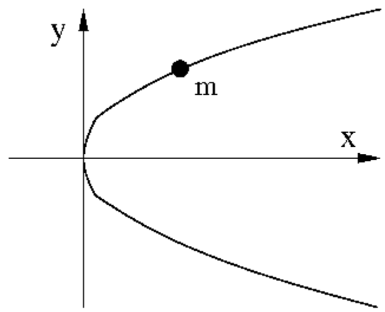

Figure 1: A particle moving under its own weight on a smooth parabola.

   Lagrangian and the corresponding Lagrange equation of motion (5 points).

   (b) Compute the corresponding generalized momentum p, write down the corresponding Hamiltonian and Hamilton equations of motion. (5 points).

   (c) Define Legendre transform and use the example to illustrate the relation between the Lagrangian and Hamiltonian involving the Legendre transform (5 points).

   (d) Use the example to illustrate the equivalence of Lagrangian and Hamiltonian formalisms (5 points).

Answers:

(a)

$$\begin{cases} x = q^2 \\ y = q \end{cases} \quad \Longrightarrow \quad \begin{cases} x = 2q\dot{q} \\ y = \dot{q} \end{cases}$$

gives the Lagrangian:

$$L = T - V = \frac{1}{2}m(4q^2 + 1)\dot{q}^2 - mgq$$

The Langrange equation of motion,

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} = 0$$

is as follows:

$$-[m(4q^2 + 1)\dot{q}]^{\cdot} + (4mq\dot{q}^2 - mg) = 0$$

(b) Generalized momentum:

$$p := \frac{\partial T}{\partial \dot{q}} = m(4q^2+1)\dot{q} \qquad \Longrightarrow \qquad \dot{q} = \frac{p}{m(4q^2+1)}$$

The Hamiltonian:

$$H = T + V = \frac{p^2}{2m(4q^2+1)} + mgq$$

The Hamilton equations of motion:

$$\begin{cases} \dot{p} = & -\dfrac{\partial H}{\partial q} \\ \dot{q} = & \dfrac{\partial H}{\partial p} \end{cases}$$

look as follows:

$$\begin{cases} \dot{p} = & \dfrac{4p^2 q}{m(4q^2+1)^2} - mg \\ \dot{q} = & \dfrac{p}{m(4q^2+1)} \end{cases}$$

(c) See the class notes.

(d) Solve the second Hamilton equation for p and plug into the first one to obtain the Lagrange equation.

2. A 2D elasticity problem.

(a) Write down the complete system of two-dimensional equations for static linear elasticity in terms of displacement components u_i, linearized strains ϵ_{ij} and Cauchy stress σ_{ij}. Explain the origin and meaning of each group of equations (8 points)

(b) Write down and discuss the Cauchy relation between stresses σ_{ij}, stress vector t_i and a normal vector $\boldsymbol{n}$ (5 points).

(c) Assume that the plate shown in Fig. 2 is uniformly stretched in x_1 direction:

$$\begin{cases} u_1 = cx_1 \\ u_2 = 0 \end{cases}$$

Use the equations of elasticity to compute the corresponding strain, stress, body forces within the plate and traction vector (stress vector) on the boundary of the plate, in terms of constant c and Lamé constants μ, λ. Draw the tractions (the load applied to the plate causing the deformation) (7 points).

Answers:

(a) See the book and class notes.

(b) See the book and class notes.

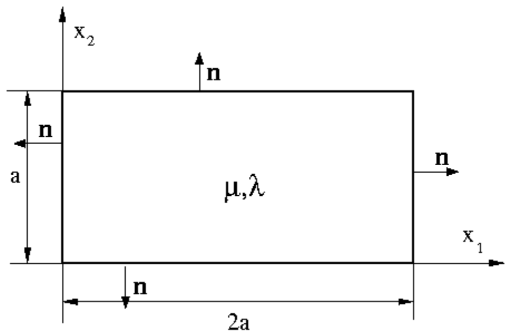

Figure 2: An elastic plate

(c) Linearized strain:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} = c, \quad \epsilon_{12} = \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) = 0, \quad \epsilon_{22} = \frac{\partial u_2}{\partial x_2} = 0$$

Use Hooke's law:

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\epsilon_{kk}\delta_{ij}$$

to compute the corresponding stresses:

$$\sigma_{11} = (2\mu + \lambda)c, \quad \sigma_{12} = 0, \quad \sigma_{22} = \lambda c$$

Use the Cauchy relation $t_i = \sigma_{ij}n_j$ to compute the tractions on the plate boundary;

$$\begin{array}{lll} x_1 = 2a & \boldsymbol{n} = (1,0) & \boldsymbol{t} = (\sigma_{11}, \sigma_{12}) = ((2\mu+\lambda)c, 0) \\ x_2 = a & \boldsymbol{n} = (0,1) & \boldsymbol{t} = (\sigma_{12}, \sigma_{22}) = (0, \lambda c) \\ x_1 = 0 & \boldsymbol{n} = (-1,0) & \boldsymbol{t} = (-\sigma_{11}, -\sigma_{12}) = (-(2\mu+\lambda)c, 0) \\ x_2 = 0 & \boldsymbol{n} = (0,-1) & \boldsymbol{t} = (-\sigma_{12}, -\sigma_{22}) = (0, -\lambda c) \end{array}$$

See Fig. 3 for the illustration of the load. Stresses are constant so the body forces vanish.

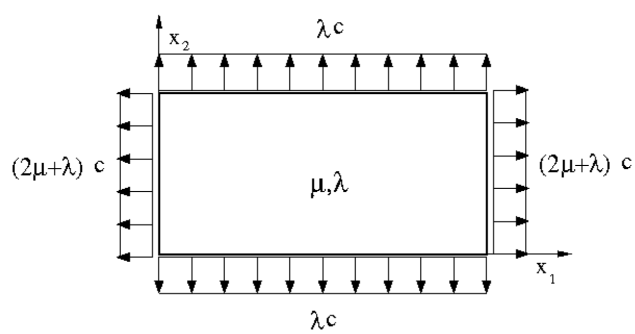

Figure 3: A uniformly stretched elastic plate

3. Consider two infinitely long straight conductors (cables) shown in Fig. 4.

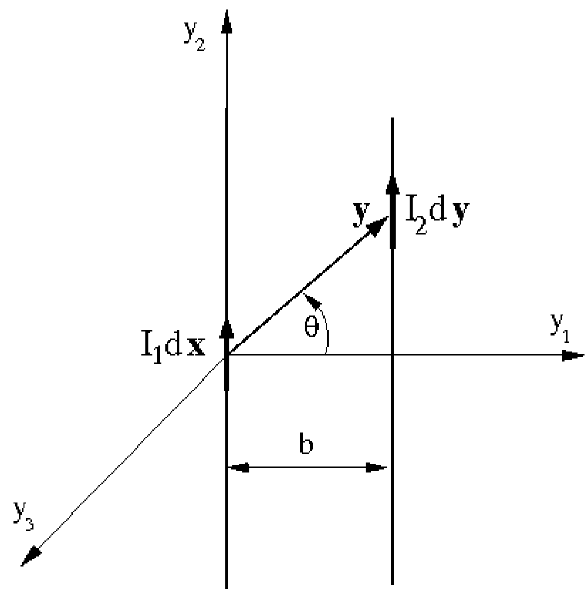

Figure 4: Parallel conductors

(a) State the Ampère's force law (5 points).

(b) Use the law to compute the force *per unit length* exerted by (the whole) conductor carrying I_2 on conductor carrying I_1, *Hint:* Integrate first in y_2 and then switch to θ shown in the figure (15 points).

Answers:

(a) Consider two current elements illustrated in Fig. 5. The force exerted on current element $I_1 d\boldsymbol{x}$

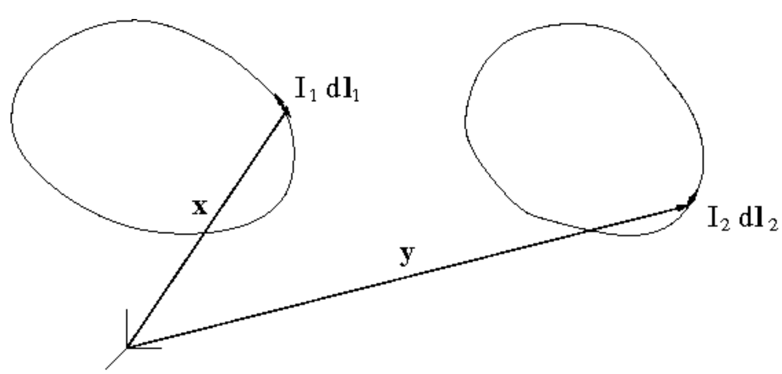

Figure 5: Ampère's Force Law

by current element $I_2 d\boldsymbol{y}$ is given by:

$$d\boldsymbol{F}_{12} = \frac{\mu_0}{4\pi} \frac{I_1 d\boldsymbol{x} \times \left(I_2 d\boldsymbol{y} \times \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|} \right)}{|\boldsymbol{x}-\boldsymbol{y}|^2}$$

where μ_0 is the *free space permeability*,

$$\mu_0 = 4\pi 10^{-7} \quad [\frac{\mathrm{N}}{\mathrm{A}^2} = \frac{\mathrm{h}}{\mathrm{m}}]$$

(b) The only non-zero component is the x_1 component and it is equal to:

$$\frac{\mu_0}{4\pi} I_1 I_2 b \int_{-\infty}^{\infty} \frac{dy_2}{(b^2 + y_2^2)^{\frac{3}{2}}} = \frac{\mu_0}{4\pi b} I_1 I_2 \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta = \frac{\mu_0}{2\pi b} I_1 I_2$$

4. Consider the infinite well problem, i.e.

$$V(x) = \begin{cases} 0 & x \in (0, a) \\ \infty & \text{otherwise} \end{cases}$$

   (a) Reproduce the reasoning from the class and determine stationary wave functions $\psi_n(x)$ and the corresponding energies E_n. Utilize the notation:

$$\omega = \frac{\pi^2 \hbar}{2ma^2}$$

   (15 points).

   (b) Assume that the initial condition for a particle in the well is given by a mixture of the first two stationary states:

$$\Psi(x, 0) = A(\psi_1(x) + \psi_2(x))$$

   Determine A and the corresponding wave function $\Psi(x, t)$ (10 points).

   (c) Find the expectation value of Hamiltonian (total energy) H. Compare it with energies E_1, E_2 corresponding to the first two states (5 points).

   (d) According to the *generalized statistical interpretation*, if you measure the energy of the particle, what do you expect to see ? (10 points)

Answers:

   (a) See class notes for details.

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a}, \quad E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2} = n^2 \omega \hbar, \quad n = 1, 2, \dots$$

   (b) Normalizing the wave function, we get $A = 1/\sqrt{2}$. This gives

$$\Psi(x, t) = \frac{1}{\sqrt{2}} \left[\psi_1(x) e^{-i\omega t} + \psi_2(x) e^{-i4\omega t}\right]$$

   (c) The expected value of total energy is between E_1 and E_2.

$$\begin{aligned}
\langle H \rangle &= \frac{1}{2} \int_0^a (\psi_1 e^{i\omega t} + \psi_2 e^{i4\omega t}) H (\psi_1 e^{-i\omega t} + \psi_2 e^{-i4\omega t}) \, dx \\
& \frac{1}{2} \int_0^a (\psi_1 e^{i\omega t} + \psi_2 e^{i4\omega t})(\omega \hbar \psi_1 e^{-i\omega t} + 4\omega \hbar \psi_2 e^{-i4\omega t}) \, dx \\
&= \frac{1}{2} \left(\omega \hbar \int_0^a |\psi_1|^2 \, dx + 4\omega \hbar \int_0^a |\psi_2|^2 \, dx + \int_0^a \psi_1 \psi_2 \, dx [4\omega \hbar e^{-i3t} + \omega \hbar e^{i3t}] \right) \\
&= \frac{5}{2} \omega \hbar
\end{aligned}$$

   (d) We do not expect to see computed value. We expect to see either E_1 or E_2, each with the same probability of $1/2$.