Linearly Stable Relative Equilibria Utilizing a Dominant Mass

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N-body Problem

Given n bodies in space with initial positions and velocities, under mutual gravitational attraction, what is their future motion?
Define:

m_i = mass of the i-th body $\qquad \mathbf{q}_i$ = position of the i-th body in $\mathbb{R}^2$

$F = ma$ gives,

$$m_i\ddot{\mathbf{q}}_i = \sum_{j\neq i}^{n} \frac{m_i m_j(\mathbf{q}_j - \mathbf{q}_i)}{||\mathbf{q}_i - \mathbf{q}_j||^3} \qquad \text{for each } i \in (1, 2, ..., n)$$

Central Configurations

A *central configuration* (c.c.) is an initial configuration of bodies $\mathbf{x}_i$, which satisfies the following equation

$$\sum_{j\neq i}^{n} \frac{m_i m_j(\mathbf{x}_j - \mathbf{x}_i)}{||\mathbf{x}_j - \mathbf{x}_i||^3} + \omega^2 m_i \mathbf{x}_i = 0 \qquad \forall i \in (1 \dots n)$$

for some proportionality constant ω^2.

- Providing a c.c. with the correct choice of initial velocities leads to a periodic solution called a relative equilibrium.
- Released from rest, a c.c. maintains the same shape as it heads toward total collision.
- Any Kepler orbit can be attached to a c.c. to obtain a new solution to the n-body problem.
- **Key Fact**: Central configurations are critical points of the Newtonian potential function U restricted to the level surface $I = I_0$.

  where $I = \frac{1}{2}\sum_{i=1}^{n} m_i||\mathbf{x}_i||^2$

Linear Stability

Due to symmetry, there are two invariant subspaces W_1 and W_2 that lead to the eight eigenvalues $0, 0, \pm\omega i, \pm\omega i, \pm\omega i$, for any relative equilibrium x. We say $\mathbf{x}$ is *nondegenerate* if the remaining $4n-8$ eigenvalues are nonzero. It is *spectrally stable* if the eigenvalues are pure imaginary and is *linearly stable* if in addition, the restriction of the matrix obtained upon linearizing about x, to the skew-orthogonal complement of $W_1 \cup W_2$ is diagonalizable.

Known Results

- In 1772, Lagrange discovered that three bodies of any mass located at the vertices of an equilateral triangle represent a solution to the 3-body problem. In 1843 Gascheau (and later in 1875 Routh) determined stability for Lagrange's solution iff

$$\frac{m_1m_2 + m_1m_3 + m_2m_3}{(m_1+m_2+m_3)^2} < \frac{1}{27}$$

- All collinear relative equilibria are unstable

Goal

We want to determine if linearly stability in a relative equilibrium implies the existence of a dominant mass in the configuration. In other words, given a linearly stable relative equilibrium, we want to find the smallest possible ratio between the dominant mass and the sum of the remaining masses, for various n values. (*Rick Moeckel*)

- We normalize the sum of the non-dominant masses to 1: $\sum_{i=2}^{n+1} m_i = 1$.
- Let $R = \frac{m_1}{m_2+m_3+...+m_{n+1}}$, and assume $m_1 > m_2 > ... > m_{n+1}$

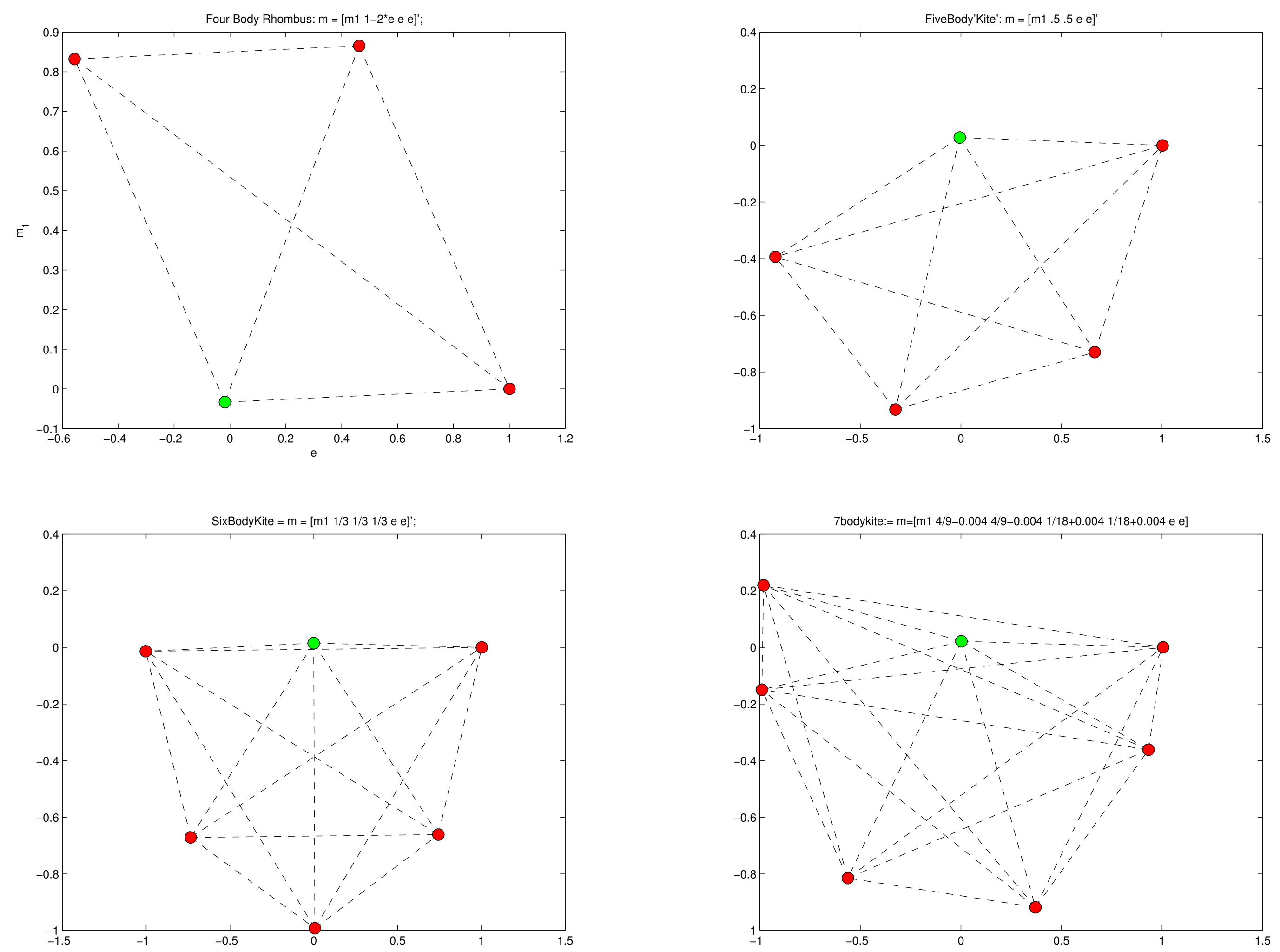
  We want to find the infimum of R for different n values in the $(1+n)$-body problem

Results

n	R
2	24.9599
3	24.9599
4	25.5132
5	31.5595
6	36.9582
7	19.9789
8	26.5326
9	33.7930
10	42.0993

Note: For $n = 3$ and $n = 4$ the analytic results are, $m_1 = \frac{25+3\sqrt{69}}{2} \approx 24.9599$

Conjecture: We want the limiting configuration to have bodies that don't collide, and we expect symmetry to lead to the best result



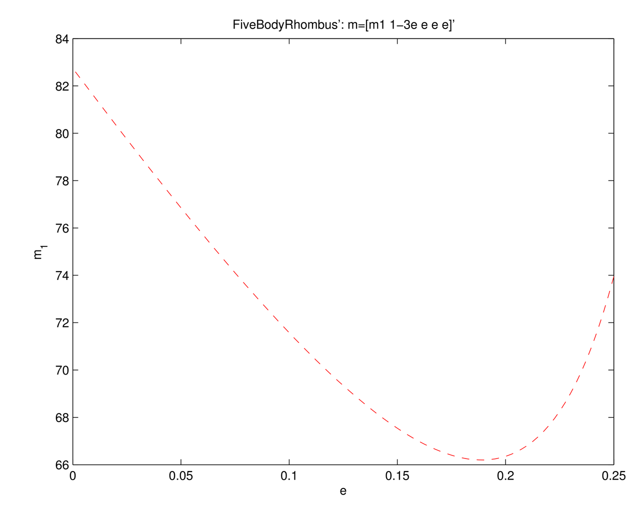
Acknowledgments

- Thanks to the Holy Cross Summer Research Program
- Thanks to the National Science Foundation, award DMS-1211675.

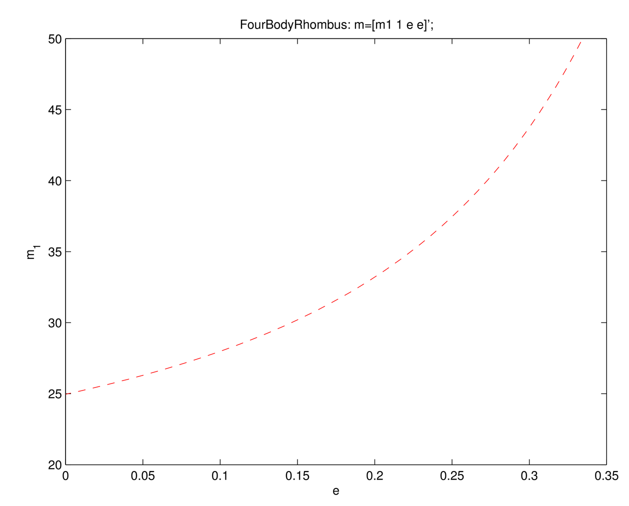
Techniques: MatLab

- We use gradient flow to locate minima of the Newtonian potential function U constrained to a level surface of the moment of inertia $I = \frac{1}{2}$.
- Given an initial configuration $\mathbf{x}_0$ and a set of masses m, a MatLab function called config_analyzer(x_0, m) was written to find a local minimum and determine its stability.
- Numerics: ODE15s (stiff ODE), RelTol, AbsTol = 1e-10, and $||M^{-1}\nabla U(\mathbf{x}) + \omega^2\mathbf{x}||$ is always $\leq$ 1e-13. Check if real parts of eigenvalues are 0 (within 1e-10).
- For each family of relative equilibria, we use a "binary search" method incorporating *config_analyzer()* to locate the smallest possible m_1 such that the corresponding relative equilibrium is linearly stable.

Clustering



Clustering example



Non-clustering example

Conjecture: Let $m_i, m_j = \epsilon$. If $\lim_{\epsilon\to 0} r_{ij} = 0$, then the dominant mass required to keep the relative equilibrium linearly stable increases as $\epsilon \to 0$.

$1+n$-gon

- For $n \geq 7$ the $(1+n)$-gon is a linearly stable relative equilibrium iff the central mass is at least $0.435n^3$ (Roberts 2000) (Moeckel 1994).