Given $n$ bodies in space with initial positions and velocities, under mutual gravitational attraction, what is their future motion? Define:

$$m_i = \text{mass of the } i\text{-th body} \quad q_i = \text{position of the } i\text{-th body in } \mathbb{R}^2$$

$$F = ma$$

Then, we have:

$$m_i q_i = \sum_{j \neq i} \frac{m_i m_j (q_i - q_j)}{|q_i - q_j|^3} \quad \text{for each } i \in \{1, 2, \ldots, n\}$$

Central Configurations

A central configuration (c.c.) is an initial configuration of bodies $x_i$, which satisfies the following equation:

$$\sum_{j \neq i} \frac{m_i m_j (x_i - x_j)}{|x_i - x_j|^3} + \omega^2 m_i x_i = 0 \quad \forall i \in \{1, 2, \ldots, n\}$$

for some proportionality constant $\omega^2$.

- Providing a c.c. with the correct choice of initial velocities leads to a periodic solution called a relative equilibrium.
- Released from rest, a c.c. maintains the same shape as it heads toward total collision.
- Any Kepler orbit can be attached to a c.c. to obtain a new solution to the $n$-body problem.

Key Fact: Central configurations are critical points of the Newtonian potential function $U$ restricted to the level surface $I = I_0$,

where $I = \frac{1}{2} \sum_{i=1}^{n} m_i ||x_i||^2$

Linear Stability

Due to symmetry, there are two invariant subspaces $W_1$ and $W_2$ that lead to the eight eigenvalues 0, 0, $\pm w_2$, $\pm w_3$, $\pm w_4$, for any relative equilibrium $x$. We say $x$ is nondegenerate if the remaining $n - 8$ eigenvalues are nonzero. It is spectrally stable if the eigenvalues are pure imaginary and is linearly stable if in addition, the restriction of the matrix obtained upon linearizing about $x$, to the skew-orthogonal complement of $W_1 \cup W_2$ is diagonalizable.

Known Results

- In 1772, Lagrange discovered that three bodies of any mass located at the vertices of an equilateral triangle represent a solution to the 3-body problem. In 1843, Gascheau (and later in 1875, Routh) determined stability for Lagrange’s solution if

$$\frac{m_1 m_2 + m_1 m_3 + m_2 m_3}{(m_1 + m_2 + m_3)^2} < \frac{1}{27}$$

- All collinear relative equilibria are unstable

N-body Problem

We want to determine if linear stability in a relative equilibrium implies the existence of a dominant mass in the configuration. In other words, given a linearly stable relative equilibrium, we want to find the smallest possible ratio between the dominant mass and the sum of the remaining masses, for various $n$ values. (Rick Moeckel)

- We normalize the sum of the non-dominant masses to 1:

$$\sum_{i=1}^{n} m_i = 1$$

- Let $R = \frac{m_1}{m_2 + m_3 + \cdots + m_n}$, and assume $m_1 > m_2 > \cdots > m_n$.

We want to find the infimum of $R$ for different $n$ values in the $(1+n)$-body problem:

$$n \quad R \quad 24.9599 \quad \text{Note: For } n = 3 \text{ and } n = 4 \text{ the analytic results are, } m_1 = \frac{25 + 3\sqrt{65}}{3} \approx 24.9599$$

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Conjecture: Let $m_1, m_j = \epsilon$. If $\lim_{\epsilon \to 0} R_{ij} = 0$, then the dominant mass required to keep the relative equilibrium linearly stable increases as $\epsilon \to 0$.

Techniques: MatLab

- We use gradient flow to locate minima of the Newtonian potential function $U$ constrained to a level surface of the moment of inertia $I = \frac{1}{2}$.
- Given an initial configuration $x_0$ and a set of masses $m$, a MatLab function called $\text{config\_analyzer}(x_0, m)$ was written to find a local minimum and determine its stability.
- Numerics: ODE15s ( stiff ODE), Ret Tol, Abs Tol = 1e-10, and $||M^{-1/2}U(x) + \omega^2 x||$ is always $\leq 1e-13$. Check if real parts of eigenvalues are 0 (within 1e-10).
- For each family of relative equilibria, we use a “binary search” method incorporating $\text{config\_analyzer}(\cdot)$ to locate the smallest possible $m_1$ such that the corresponding relative equilibrium is linearly stable.

Clustering

Conjecture: Let $m_1, m_j = \epsilon$. If $\lim_{\epsilon \to 0} R_{ij} = 0$, then the dominant mass required to keep the relative equilibrium linearly stable increases as $\epsilon \to 0$. For $n \geq 7$ the $(1+n)$-gon is a linearly stable relative equilibrium iff the central mass is at least $0.435 m^2$ (Roberts 2000) (Moeckel 1994).

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