We consider Euler’s equation, which models the motion of inviscid fluids. Given \( u = \text{velocity}, \ p = \text{pressure}, \ \rho = \text{density} \) we have 
\[
\frac{\partial u}{\partial t} + u \cdot \nabla u + \frac{1}{\rho} \nabla p = 0
\]
Determining if solutions to the three dimensional Euler’s equation exist globally in time or blow up in finite time is an active area of research. Blow up occurs when the gradient of the velocity becomes infinitely large. Beale, Kato, and Majda (1984) showed that this is equivalent to the curl of the velocity, the vorticity, tending towards infinity.

To make the problem more tractable, Constantin, Lax and Majda (1985) showed that this is equivalent to the curl of the velocity, the vorticity, tending towards infinity.

Gregorio (1989) modified this equation by adding a convection term, \( a < \gamma \) as we increase \( a \) values will converge to some fixed at \( \gamma \). Okamoto (2008) generalized this model by placing a constant in front of \( a \) and use Matlab’s ODE45 to solve for the coefficients over time.

Finite Difference Method

We use an upwinding scheme for \( w \), Matlab’s Hilbert Transform function for \( Hw \), and the following first order approximation for \( w \):
\[
w_t = \frac{w(t + \Delta t, x) - w(t, x)}{\Delta t}
\]

Accuracy of Methods

The \( L^2 \) norm, used to measure distance between solutions, is defined below:
\[
|w_{N_1} - w_{N_2}|_2 = (\int_\pi^\pi |w_{N_1} - w_{N_2}|^2 \, dx)^{1/2}
\]

\( L^2 \) Convergence of Methods for \( a = 0.7, t_f = 2 \)

**Spectral Method**

Assuming solutions are periodic with form 
\[
w(x, t) = \sum_{n=-\infty}^{\infty} w_n(t) e^{inx},
\]
we obtain the following ODE for \( k \in [-N, N] \)
\[
w'_k(t) = -\sum_{n=-N}^{N} \text{sgn}(k-n) w_n(t) w_{k-n}(t) + a \sum_{n=-N}^{N} \frac{i n}{|k|} w_n(t) w_{k-n}(t) - \nu(\pi) \sin(k \pi) w_k(t)
\]
and use Matlab’s ODE45 to solve for the coefficients over time.

**Numerical Methods**

**Bifurcation Results**

Given a fixed final time, \( t_f \), we use a binary search method to find the largest \( a \) for which blow up occurs. Our hope is that the \( a \) values will converge to some \( \gamma \) as we increase \( t_f \), producing graphs like the following.

No Blow up of \( w(x, t) \): \( a > \gamma \)

The difficulty in approximating \( a \):

- Blow up could result from accumulation of error due to the time-step (here fixed at \( \Delta t = 10^{-4} \) or truncation in the spectral method.
- Difficult to distinguish blow up solutions from solutions growing in time.
- Observations using the binary search method indicate:
  - ODE45 warning and \( \int_0^T |H(w_{N_1})| \, dt > 10^{12} \) flags, produce the same intervals of \( a \) values where blow up occurs.

**Analytical Attempts**

- Considering Cordoba’s reformulation of the PDE:
  - Find conditions where \( H(vw) < 0 \) to prove blow up for a finite \( a > 0 \).
  - Given the generalized Ricatti equation:
    \[
y'(t) = \frac{1}{\Delta t} y(t)^2 + x(t),
\]
in order to relate Cordoba’s equation to a Ricatti form, where conditions for blow up are known.

We write solutions in the form \( w(t) = w_{N_1}(t) + H(w_{N_1})(t) \) and use a Fefferman-Stein decomposition to produce a system of ODEs in which to look for bifurcations.