A new finite element method for simulating surface plasmon polaritons on graphene sheets

Jichun Li * Li Zhu † Todd Arbogast ‡

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Abstract

In this paper, we develop a new variational form to simulate the propagation of surface plasmon polaritons on graphene sheets. Here the graphene is treated as a thin sheet of current with an effective conductivity. A novel finite element method is proposed for solving this graphene model. Discrete stability and error estimate are proved for our proposed method. Numerical results are presented to demonstrate the effectiveness of this graphene model for simulating the surface plasmon polaritons propagating on graphene sheets.

Keywords – Maxwell’s equations, finite element time-domain methods, edge elements, graphene.

Mathematics Subject Classification (2000): 78M10, 65N30, 65F10, 78-08.

1 Introduction

The two-dimensional (2-D) material graphene was rediscovered, isolated and investigated by Novoselov, Geim and co-workers [29] in 2004. The 2010 Nobel Prize in Physics was awarded to Geim and Novoselov “for groundbreaking experiments regarding the two-dimensional material graphene.” Since 2004, graphene has become a valuable and useful nanomaterial, and its study has become a very hot research topic [4,12,34] due to its exceptionally high tensile strength, high electronic mobility, high thermal conductivity, low absorption of light, and being the thinnest two-dimensional material in the world.

Numerical simulation of electromagnetic wave propagation plays a very important role in the study of graphene and its applications. The finite difference time-domain (FDTD) method (e.g., [1,11,13,14,18,23,39]) and the finite element method (FEM) (e.g., [3,6–8,10,15,16,20,30]) are arguably the two most popular numerical methods in computational electromagnetics, which

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*Department of Mathematical Sciences, University of Nevada Las Vegas, Nevada 89154-4020, USA (jichun.li@unlv.edu)
†Department of Mathematical Sciences, University of Nevada Las Vegas, Nevada 89154-4020, USA (zhul5@unlv.nevada.edu).
‡Department of Mathematics, University of Texas at Austin, Austin, Texas 78712-1202, USA (arbogast@oden.utexas.edu)
can solve Maxwell’s equations in various media. More details and references on the FDTD method and FEM for Maxwell’s equations can be found in related FDTD books [33] and FEM books [9,21,27].

Compared to many existing papers on simulation of graphene and its applications by FDTD methods [5, 26, 28], there are quite limited publications on FEMs for graphene simulation, e.g., [22, 36] are on discontinuous Galerkin time-domain (DGTD) modeling of graphene devices, and [25, 32] are on frequency-domain finite element simulation of graphene sheet. Recently, Li and collaborators [17, 19, 38] have proposed and analyzed some finite element time-domain (FETD) methods for graphene simulation. In [17, 19, 38], the graphene has been treated with some thickness (though very thin). A major drawback of this approach is that a particularly fine spatial mesh is needed for the graphene part, which makes the implementation time consuming. Mathematical analysis of graphene model in time domain is very limited. In a recent work [37], the authors investigated the effects of modulating the electronic doping of graphene in time on plasmon dynamics, and they also established the existence, uniqueness, and regularity for solutions to the resulting current equation. In this paper, we will investigate a time-domain graphene model and treat the graphene as an infinitesimal thin conductive sheet. For the first time a new finite element time-domain method is proposed and analyzed for solving this graphene model.

The rest of the paper is organized as follows. In Section 2, we first present the time-domain governing equations for modeling the surface plasmon polaritons on the graphene sheet. Then we prove an energy identity and a stability for the system of the modeling equations. In Section 3, we propose a leapfrog type scheme for solving the modeling equations, and prove the discrete stability and the optimal error estimate for our scheme. In Section 4, we present extensive numerical results to demonstrate the propagation of surface plasmon polaritons appearing on various graphene sheets. We conclude the paper in Section 5.

2 The governing equations and stability analysis

We assume that Ω is a bounded Lipschitz polygonal domain in \( \mathbb{R}^2 \) with boundary \( \partial \Omega \). In our previous works [17,19,38], we treated the graphene as a homogenized material of small thickness with an effective permittivity. Here we adopt another way to treat graphene as a thin sheet of current with an effective conductivity.

Considering that the interband conductivity is not that significant in most cases, we will ignore it in this paper. For simplicity, we consider the TE\(_z\) mode problem with electric field \( \mathbf{E} = (E_x, E_y)' \) and magnetic field \( \mathbf{H} = H_z \). From [38, (2.7)-(2.12)] (see also [17, (2.10)-(2.13)]), we have the following governing equations for simulating surface plasmon propagation on graphene:

\[
\begin{align*}
\epsilon_0 \partial_t \mathbf{E} &= \nabla \times \mathbf{H}, \quad \text{in } \Omega, \\
\mu_0 \partial_t \mathbf{H} &= -\nabla \times \mathbf{E} - \mathbf{K}_s, \quad \text{in } \Omega, \\
\tau_0 \partial_t \mathbf{J} + \mathbf{J} &= \sigma_0 \mathbf{E}, \quad \text{on } \Gamma,
\end{align*}
\]

where \( \mathbf{K}_s \) is an imposed magnetic source function, \( \mathbf{J} := J_d \) (as denoted in [38]) is the induced intraband surface current in graphene, \( \epsilon_0 \) and \( \mu_0 \) are respectively the permittivity and perme-
ability in vacuum, the positive constant $\tau_0$ denotes the relaxation time, and the positive constant $\sigma_0$ denotes the graphene surface conductivity. Here $\Gamma$ represents the graphene sheet buried in the domain $\Omega$. It appears as a line in our 2D domain (cf. Figures 1, 3, 5, and 7 shown later). Finally, the 2D curl operators are defined as $\nabla \times H := (\partial_y H, -\partial_x H)'$ and $\nabla \times E := \partial_x E_y - \partial_y E_x$.

According to [2, Fig.1], the boundary conditions on the graphene interface are:

$$\hat{n}_1 \times E_1 = \hat{n}_2 \times E_2, \quad \text{on } \Gamma,$$  
(2.4)
$$H_1 - H_2 = J \times \hat{n}, \quad \text{on } \Gamma,$$  
(2.5)

which mean that the tangential electric field is continuous across the interface, and the jump of the tangential component of the magnetic field along the interface is equal to the surface current. Here $H_1$ and $H_2$ represent the magnetic field above and below the interface, respective, $\hat{n} := (n_x, n_y)'$ is the unit normal vector pointing upward, and $\hat{n}_1$ and $\hat{n}_2$ are the unit outward normal vectors from top and bottom subdomains of the interface. Here we denote the 2D cross product $J \times \hat{n} := J_x n_y - J_y n_x$.

We remark that (2.3) was originally developed for a graphene sheet with small thickness in [38]. For an infinitely thin graphene sheet, the surface current must lie within $\Gamma$, and so the equation must be interpreted as

$$\tau_0 \partial_t (J \times \hat{n}) + J \times \hat{n} = \sigma_0 E \times \hat{n}.$$  
(2.6)

To complete the problem, we assume that (2.1)-(2.3) is subject to the perfectly conducting (PEC) boundary condition:

$$\hat{\nu} \times E = 0, \quad \text{on } \partial \Omega,$$  
(2.7)

and the initial conditions

$$E(x, 0) = E_0(x), \quad H(x, 0) = H_0(x), \quad J(x, 0)|_{\Gamma} = J_0(x)|_{\Gamma},$$  
(2.8)

where $\hat{\nu}$ is the unit outward normal vector on $\partial \Omega$, and $E_0, H_0, J_0$ are some given functions.

We want to remark that the system (2.1)-(2.8) can be used to model the propagation of the surface plasmon polaritons on graphene, which is usually embedded inside other materials such as vacuum. Moreover, the system (2.1)-(2.3) can be reduced to the standard Maxwell’s equations in vacuum by setting $J$ to be zero and ignoring (2.3).

Denote the Sobolev space $H_0(\text{curl}; \Omega) = \{ u \in (L^2(\Omega))^2 : \nabla \times u \in L^2(\Omega), \ \hat{\nu} \times u = 0 \text{ on } \partial \Omega \}$. We can easily obtain the following weak formulation: Find the solution $E \in L^2(0, T; H_0(\text{curl}; \Omega)) \cap H^1(0, T; (L^2(\Omega))^2), H \in H^1(0, T; L^2(\Omega)), J \in H^1(0, T; (L^2(\Gamma))^2)$, such that

$$\epsilon_0 (\partial_t E, \phi) = (H, \nabla \times \phi) - \langle J, \phi \rangle_{\Gamma}$$  
(2.9)
$$\mu_0 (\partial_t H, \psi) = - (\nabla \times E, \psi) - (K_s, \psi)$$  
(2.10)
$$\langle \tau_0 \partial_t J, \chi \rangle_{\Gamma} + \langle J, \chi \rangle_{\Gamma} = \langle \sigma_0 E, \chi \rangle_{\Gamma}$$  
(2.11)
hold true for any test functions $\phi \in H_0(\text{curl}; \Omega), \psi \in L^2(\Omega)$ and $\chi \in (L^2(\Gamma))^2$. To obtain (2.9), we use the integration by parts over $\Omega$ and the boundary condition (2.5). Here and below we denote $\langle \cdot, \cdot \rangle$ for the inner product over $\Omega$, and $\langle J, \phi \rangle_{\Gamma} := \int_{\Gamma} J \times \hat{n} \cdot \phi \times \hat{n} \ ds$ for the inner product on $\Gamma$. Only $J \times \hat{n}$ is determined by the differential and variational formulations, and only the component $\chi \times \hat{n}$ of $\chi$ is used as a test function.

To simplify the notation, we denote the $L^2$ norm of $u$ in $\Omega$ as $||u|| := ||u||_{L^2(\Omega)}$, and the $L^2$ norm of $u$ on $\Gamma$ as $||u||_{\Gamma} := (\int_{\Gamma} |u \times \hat{n}|^2 \ ds)^{1/2}$.

**Theorem 2.1.** For the solution $(E, H, J)$ of (2.9)-(2.11), the following energy identity holds true for any $t \in [0, T]$:

$$
\text{ENG}(t) - \text{ENG}(0) + \int_{0}^{t} \frac{2}{\sigma_0} ||J||_\Gamma^2 \ dt = - \int_{0}^{t} 2(K_s, H) dt,
$$

where we denote the energy

$$
\text{ENG}(t) := \left[ \epsilon_0 ||E||^2 + \mu_0 ||H||^2 + \frac{\sigma_0}{\epsilon_0} ||J||^2_\Gamma \right](t).
$$

Furthermore, we have the following continuous stability:

$$
\text{ENG}(t) \leq \left[ \text{ENG}(0) + \int_{0}^{t} \frac{1}{\mu_0} ||K_s||^2 \ dt \right] \cdot \exp(t), \quad \forall \ t \in [0, T].
$$

**Proof.** By choosing $\phi = 2E, \psi = 2H, \chi = \frac{2}{\epsilon_0} J$ in (2.9)-(2.11), respectively, then adding the results together, we have

$$
\frac{d}{dt} \left( \epsilon_0 ||E||^2 + \mu_0 ||H||^2 + \frac{\sigma_0}{\epsilon_0} ||J||^2_\Gamma \right) + \frac{2}{\sigma_0} ||J||^2_\Gamma = -2(K_s, H).
$$

Integrating (2.15) with respect to $t$ from 0 to $t$, and using the energy notation defined by (2.13), we immediately have the energy identity (2.12).

Using the following Cauchy-Schwarz inequality

$$
- \int_{0}^{t} 2(K_s, H) dt \leq \int_{0}^{t} (\mu_0 ||H||^2 + \frac{1}{\mu_0} ||K_s||^2) dt,
$$

in (2.12), and dropping the nonnegative term $\int_{0}^{t} \frac{\sigma_0}{\epsilon_0} ||J||^2_\Gamma \ dt$ on the left hand side, we obtain

$$
\text{ENG}(t) \leq \left[ \text{ENG}(0) + \int_{0}^{t} \frac{1}{\mu_0} ||K_s||^2 \ dt \right] + \int_{0}^{t} \mu_0 ||H||^2 \ dt
\leq \left[ \text{ENG}(0) + \int_{0}^{t} \frac{1}{\mu_0} ||K_s||^2 \ dt \right] + \int_{0}^{t} \text{ENG}(s) ds.
$$

The proof of (2.14) is completed by the Gronwall inequality applied to (2.16). $\square$

### 3 The leapfrog finite element scheme and its analysis

To design a finite element method, we partition the physical domain $\Omega$ with $\Gamma$ as an internal boundary by a shape regular triangular mesh $T_h$ with maximum mesh size $h$. Without loss of
generality, we consider the following Raviart-Thomas-Nédélec (RTN)’s mixed spaces $U_h$ and $V_h$ on triangular elements [21,27]: For any $r \geq 1$,
\[ U_h = \{ u_h \in L^2(\Omega) : u_h|_K \in p_{r-1}, \forall K \in T_h \}, \]
\[ V_h = \{ v_h \in H(curl; \Omega) : v_h|_K \in (p_{r-1})^2 \oplus S_r, \forall K \in T_h \}, \quad S_r = \{ \vec{p} \in p_r^2, \vec{x} \cdot \vec{p} = 0 \}. \]
To handle the PEC boundary condition (2.7), we introduce the subspace
\[ V_h^0 = \{ v_h \in V_h : \hat{\nu} \times v_h = 0 \text{ on } \partial \Omega \} . \]
To construct the fully discrete finite element scheme, we partition the time interval $[0,T]$ uniformly by points $t_i = i\tau, i = 0, ..., N_t$, where $\tau = \frac{T}{N_t}$ denotes the time step size.
Now we can construct the following leapfrog type scheme: Given proper initial approximations of $E_h^0 \in V_h^0, J_h^\frac{1}{2} \in W_h, H_h^\frac{1}{2} \in U_h$, for any $n \geq 0$, find $E_h^{n+1} \in V_h^0, J_h^{n+\frac{1}{2}} \in W_h$ (i.e., $J_h^{n+\frac{1}{2}} \times \hat{n} \in W_h \times \hat{n}$), $H_h^{n+\frac{3}{2}} \in U_h$ such that
\begin{align*}
\epsilon_0 (\partial_{\tau} E_h^{n+\frac{1}{2}}, \phi_h) &= (H_h^{n+\frac{1}{2}}, \nabla \times \phi_h) - (J_h^{n+\frac{1}{2}}, \phi_h)\Gamma, \quad \text{(3.1)} \\
\mu_0 (\partial_{\tau} H_h^{n+1}, \psi_h) &= - (\nabla \times E_h^{n+1}, \psi_h) - (K_\nu^{n+1}, \psi_h), \quad \text{(3.2)} \\
\tau_0 (\partial_{\tau} J_h^{n+1}, \chi_h)\Gamma + (J_h^{n+1}, \chi_h)\Gamma &= \sigma_0 (E_h^{n+1}, \chi_h)\Gamma, \quad \text{(3.3)}
\end{align*}
hold true for any test functions $\phi_h \in V_h^0, \psi_h \in U_h$ and $\chi_h \in W_h$. Here we choose
\[ W_h = \{ w_h \in (L^2(\Omega))^2 : \exists v_h \in V_h \text{ such that } v_h \times \hat{n} = w_h \text{ on } \Gamma \}, \]
and adopt the following central difference operator and averaging operator in time: For any time sequence function $u^n$,
\[ \delta _\tau u^{n+\frac{1}{2}} = \frac{u^{n+1} - u^n}{\tau}, \quad \bar{u}^{n+\frac{1}{2}} = \frac{u^{n+1} + u^n}{2}. \]
Corresponding to the finite element spaces $V_h$ and $U_h$, we denote $\Pi_1$ and $\Pi_2$ for the standard Nédélec interpolation in space $V_h$ and the standard $L^2$ projection onto space $U_h$, respectively.
Furthermore, the following interpolation and projection errors hold true (cf. [21,27]):
\begin{align*}
||u - \Pi_1 u||_{H^r(\text{curl}; \Omega)} &\leq c h^r ||u||_{H^r(\text{curl}; \Omega)}, \quad \forall u \in H^r(\text{curl}; \Omega), \quad r \geq 1, \quad \text{(3.4)} \\
||u - \Pi_2 u||_{L^2(\Omega)} &\leq c h^r ||u||_{H^r(\Omega)}, \quad \forall u \in H^r(\Omega), \quad r \geq 0, \quad \text{(3.5)}
\end{align*}
where $||u||_{H^r(\Omega)}$ denotes the norm for the Sobolev space $H^r(\Omega)$, and $||u||_{H^r(\text{curl}; \Omega)} := (||u||_{H^r(\Omega)})^2 + ||\nabla \times u||_{H^r(\Omega)}^2 + ||\nabla \times u||_{H^r(\Omega)}^{1/2}$ is the norm for the Sobolev space $H^r(\text{curl}; \Omega) = \{ u \in (H^r(\Omega))^2 : \nabla \times u \in H^r(\Omega) \}$.

The initial conditions (2.8) are discretized as follows:
\begin{align*}
E_h^0 &= \Pi_1 E_0(x), \quad \text{(3.6)} \\
H_h^\frac{1}{2} &= \Pi_2 (H(\cdot, 0) + \frac{\tau}{2} \partial_{\tau} H(\cdot, 0)) = \Pi_2 \left[ H_0(x) - \frac{\tau}{2\mu_0} (\nabla \times E_0(x) + K_s(x, 0)) \right], \quad \text{(3.7)} \\
J_h^{\frac{1}{2}} \times \hat{n} &= \Pi_2 \left[ (J(\cdot, 0) + \frac{\tau}{2} \partial_{\tau} J(\cdot, 0)) \times \hat{n} \right] = \Pi_2 \left[ J_0 \times \hat{n} + \frac{\tau}{2\tau_0} (\sigma_0 E_0 - J_0) \times \hat{n} \right], \quad \text{(3.8)}
\end{align*}
where we use the Taylor expansion and the governing equations (2.2) and (2.3).
Below we will present the stability and convergence analysis for our scheme.
3.1 Stability analysis

To prove the discrete stability for the fully-discrete scheme, we denote the wave propagation speed in vacuum $C_v = \frac{1}{\sqrt{\mu_0}} \approx 3 \times 10^8 \text{m/s}$, and introduce the standard inverse estimate:

$$\|\nabla \times \phi_h\| \leq C_{in} h^{-1} \|\phi_h\|, \quad \forall \phi_h \in V_h,$$

(3.9)
and the trace estimate:

$$\|\phi_h\|_{L^2(\Gamma)} \leq C_{tr} h^{-1/2} \|\phi_h\|_{L^2(\Omega)}, \quad \forall \phi_h \in V_h,$$

(3.10)
where the positive constants $C_{in}$ and $C_{tr}$ are independent of the mesh size $h$.

**Theorem 3.1.** Denote the discrete energy:

$$ENG \text{dis}(m) := \epsilon_0 \|E_h^m\|^2 + \mu_0 \|H_h^{m+\frac{1}{2}}\|^2 + \frac{\sigma_0}{\sigma} \|J_h^{m+\frac{1}{2}}\|^2.$$  

(3.11)

Then under the time step constraint:

$$\tau \leq \min \left( \frac{1}{2}, \frac{h}{2C_v C_{in}}, \frac{h^2}{2C_{tr}} \sqrt{\frac{\epsilon_0 \sigma_0}{\sigma_0}} \right),$$

(3.12)
we have: For any $m \in [1, N_t - 1]$,

$$ENG \text{dis}(m) \leq 2 \left[ ENG \text{dis}(0) + \tau \sum_{n=0}^{m-1} \frac{1}{\mu_0} \|K_{s}^{n+1}\|^2 \right] \exp(2m\tau).$$

(3.13)

**Proof.** Choosing $\phi_h = 2\tau E_h^{n+\frac{1}{2}}, \psi_h = 2\tau H_h^{n+\frac{1}{2}}, \chi_h = \frac{2\tau}{\sigma_0} \mathcal{J}_{h}^{n+1}$ in (3.1)-(3.3), respectively, then adding them together, we have

$$\epsilon_0 (\|E_h^{n+1}\|^2 - \|E_h^n\|^2) + \mu_0 (\|H_h^{n+\frac{1}{2}}\|^2 - \|H_h^{n+\frac{1}{2}}\|^2) + \frac{\sigma_0}{\sigma} (\|J_h^{n+\frac{1}{2}}\|^2 - \|J_h^{n+\frac{1}{2}}\|^2)$$

$$+ \frac{2\tau}{\sigma_0} \|\mathcal{J}_{h}^{n+1}\|^2 = \tau \left( (H_h^{n+\frac{1}{2}}, \nabla \times E_h^n) - (H_h^{n+\frac{1}{2}}, \nabla \times E_h^{n+1}) \right)$$

$$+ \tau \left( (E_h^{n+1}, J_h^{n+\frac{1}{2}})_\Gamma - (E_h^{n}, J_h^{n+\frac{1}{2}})_\Gamma \right) - 2\tau (K_{s}^{n+1}, \mathcal{H}_{h}^{n+1}).$$

(3.14)

Now summing up (3.14) over $n$ from $n = 0$ to any $m \leq N_t - 2$, and dropping the nonnegative term $\frac{2\tau}{\sigma_0} \|\mathcal{J}_{h}^{n+1}\|^2$ on the left hand side of (3.14), we obtain

$$\epsilon_0 (\|E_h^{n+1}\|^2 - \|E_h^n\|^2) + \mu_0 (\|H_h^{n+\frac{1}{2}}\|^2 - \|H_h^{n+\frac{1}{2}}\|^2) + \frac{\sigma_0}{\sigma} (\|J_h^{n+\frac{1}{2}}\|^2 - \|J_h^{n+\frac{1}{2}}\|^2)$$

$$\leq \tau \left( (H_h^{n+\frac{1}{2}}, \nabla \times E_h^{n}) - (H_h^{n+\frac{1}{2}}, \nabla \times E_h^{n+1}) \right)$$

$$+ \tau \left( (E_h^{n+1}, J_h^{n+\frac{1}{2}})_\Gamma - (E_h^{n}, J_h^{n+\frac{1}{2}})_\Gamma \right) - 2\tau \sum_{n=0}^{m} (K_{s}^{n+1}, \mathcal{H}_{h}^{n+1}).$$

(3.15)

By the inverse estimate (3.9), the Cauchy-Schwarz inequality, and the notation $C_v$, we have

$$\tau (H_h^{n+\frac{1}{2}}, \nabla \times E_h^{m+1}) \leq \tau C_v \sqrt{\mu_0} \|H_h^{m+\frac{1}{2}}\| \cdot C_{in} h^{-1} \sqrt{\epsilon_0} \|E_h^{m+1}\|$$

$$\leq \frac{1}{2} \tau C_v C_{in} h^{-1} (\mu_0 \|H_h^{m+\frac{1}{2}}\|^2 + \epsilon_0 \|E_h^{m+1}\|^2),$$

(3.16)
which also holds true for \( m = -1 \).

Similarly, by the trace estimate (3.10) and the Cauchy-Schwarz inequality, we have
\[
\tau \langle E_h^{m+\frac{1}{2}}, J_{h}^{m+\frac{1}{2}} \rangle_{TV} \leq \frac{1}{2} \tau C_{tr} h^{-\frac{1}{2}} \sqrt{\frac{\sigma_0}{\epsilon_0 \tau_0} \cdot \sqrt{\epsilon_0} \| E_h^{m+1} \| \cdot \sqrt{\frac{\tau_0}{\sigma_0} \| J_h^{m+\frac{1}{2}} \|^2}}
\]
which also holds true for \( m = -1 \).

Finally, by the similar technique, we have
\[
2\tau \sum_{n=0}^{m} (K_n^{n+1}, H_n^{n+1}) \leq \tau \sum_{n=0}^{m} \left( \mu_0 \| H_h^{n+\frac{1}{2}} \|^2 + \frac{1}{\mu_0} \| K_n^{n+1} \|^2 \right)
\]
\[
\leq \frac{\tau \mu_0}{2} \| H_h^{n+\frac{1}{2}} \|^2 + \tau \sum_{n=0}^{m} \mu_0 \| H_h^{n+\frac{1}{2}} \|^2 + \tau \sum_{n=0}^{m} \frac{1}{\mu_0} \| K_n^{n+1} \|^2.
\]

Substituting the above estimates (3.16)-(3.18) into (3.15), and choosing \( \tau \) small enough, such as
\[
\tau \leq \frac{1}{2}, \quad \tau C_{tr} h^{-\frac{1}{2}} \leq \frac{1}{2}, \quad \tau C_{tr} h^{-\frac{1}{2}} \sqrt{\frac{\sigma_0}{\epsilon_0 \tau_0}} \leq \frac{1}{2},
\]
which is equivalent to (3.12), we obtain
\[
\frac{1}{2} \left( \epsilon_0 \| E_h^{m+1} \|^2 + \mu_0 \| H_h^{m+\frac{1}{2}} \|^2 + \frac{\tau_0}{\sigma_0} \| J_h^{m+\frac{1}{2}} \|^2 \right)
\]
\[
\leq \epsilon_0 \| E_h^{0} \|^2 + \mu_0 \| H_h^{\frac{1}{2}} \|^2 + \frac{\tau_0}{\sigma_0} \| J_h^{\frac{1}{2}} \|^2 + \tau \sum_{n=0}^{m} \frac{1}{\mu_0} \| K_n^{n+1} \|^2 + \tau \sum_{n=0}^{m} \mu_0 \| H_h^{n+\frac{1}{2}} \|^2.
\]

Using the discrete Gronwall inequality, we immediately have
\[
\epsilon_0 \| E_h^{m+1} \|^2 + \mu_0 \| H_h^{m+\frac{1}{2}} \|^2 + \frac{\tau_0}{\sigma_0} \| J_h^{m+\frac{1}{2}} \|^2
\]
\[
\leq 2 \left[ \epsilon_0 \| E_h^{0} \|^2 + \mu_0 \| H_h^{\frac{1}{2}} \|^2 + \frac{\tau_0}{\sigma_0} \| J_h^{\frac{1}{2}} \|^2 + \tau \sum_{n=0}^{m} \frac{1}{\mu_0} \| K_n^{n+1} \|^2 \right] \cdot \exp(2(m + 1)\tau),
\]
which completes the proof of (3.13). \( \Box \)

By Theorem 3.1, it is easy to conclude the existence of a unique solution to our scheme.

**Corollary 3.1.** Under the time constraint (3.12), for all \( n \geq 0 \), there exists a unique solution \( E_{h}^{n+1} \in V_{h}, J_{h}^{n+\frac{1}{2}} \times \hat{n} \in W_{h} \times \hat{n}, H_{h}^{n+\frac{1}{2}} \in U_{h} \) to the scheme (3.1)-(3.3).

### 3.2 Convergence analysis

To prove the error estimate for our scheme (3.1)-(3.3), we introduce the error notations:
\[
\mathcal{E}_{h}^{n} := E(t_{n}) - E_{h}^{n} = (E(t_{n}) - \Pi_{e} E(t_{n})) - (E_{h}^{n} - \Pi_{e} E(t_{n})) := E_{h}^{n} - E_{h}^{n}, \quad (3.22)
\]
\[ \mathcal{H}_h^n := H(t_n) - H^n_h = (H(t_n) - \Pi_2 H(t_n)) - (H^n_h - \Pi_2 H(t_n)) := H^n_{h\xi} - H^n_{h\eta}, \] (3.23)

where \( E^n_{h\eta}, H^n_{h\eta} \) represent the errors between the finite element solutions and the interpolations of the exact solutions, and \( E^n_{h\xi}, H^n_{h\xi} \) represent the interpolation or projection errors.

Moreover, we need the following lemma.

**Lemma 3.1.** [21, Lemmas 3.16 and 3.19] Denote \( u^n := u(\cdot, t_n) \). We have

\begin{align*}
( i ) & \quad \| \delta_t u^n + \frac{1}{\tau} \|^2 = \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \| \partial_t u(t) \|^2 dt, \quad \forall u \in H^1(0, T; L^2(\Omega)), \\
( ii ) & \quad \| \delta_t u^n + \frac{1}{\tau} \|^2 \leq \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \| \partial_t u(t) \|^2 dt, \quad \forall u \in H^2(0, T; L^2(\Omega)), \\
( iii ) & \quad \| \partial_t u^n + \frac{1}{\tau} \|^2 \leq \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \| \partial_t u(t) \|^2 dt, \quad \forall u \in H^2(0, T; L^2(\Omega)).
\end{align*}

(3.24) – (3.26)

Integrating (2.9) with \( \phi = \phi_h \) from \( t = t_n \) to \( t = t_{n+1} \), then dividing by \( \tau \), and using the result to subtract (3.1), we obtain the error equation for \( E \):

\[ e_0(\delta_t E^n_{h\eta}, \phi_h) - (H^n_{h\eta}, \nabla \phi_h) + (J^n + \frac{1}{2}, J^n_{h\eta}, \phi_h)_\Gamma, \]

\[ = \left( \frac{1}{\tau} \int_{t_n}^{t_{n+1}} H dt - H^n_{h\eta}, \nabla \phi_h \right) - \left( \frac{1}{\tau} \int_{t_n}^{t_{n+1}} J dt - J^n_{h\eta}, \phi_h \right)_\Gamma, \] (3.27)

where for simplicity we denote the exact solutions \( H^n_{h\eta} := H(\cdot, t_{n+\frac{1}{2}}) \) and \( J^n_{h\eta} := J(\cdot, t_{n+\frac{1}{2}}) \).

Using the error notations (3.22)-(3.23), we can rewrite (3.27) as follows:

\[ e_0(\delta_t E^n_{h\eta}, \phi_h) - (H^n_{h\eta}, \nabla \phi_h) + (J^n + \frac{1}{2}, J^n_{h\eta}, \phi_h)_\Gamma \]

\[ = e_0(\delta_t E^n_{h\xi}, \phi_h) - (H^n_{h\xi}, \nabla \phi_h) + (J^n + \frac{1}{2}, J^n_{h\xi}, \phi_h)_\Gamma \]

\[ + (H^n + \frac{1}{2}, \frac{1}{\tau} \int_{t_n}^{t_{n+1}} H dt, \nabla \phi_h) + \left( \frac{1}{\tau} \int_{t_n}^{t_{n+1}} J dt - J^n + \frac{1}{2}, \phi_h \right)_\Gamma, \] (3.28)

where we used the following simplified notations

\[ (J^n_{h\eta}, \phi_h)_\Gamma = \int_{\Gamma} \left( J^n_{h\eta} \times \hat{n} - \Pi_2 (J^n + \frac{1}{2} \times \hat{n}) \right) \cdot \phi_h \times \hat{n} ds, \] (3.29)

\[ (J^n_{h\xi}, \phi_h)_\Gamma = \int_{\Gamma} \left( J^n_{h\xi} \times \hat{n} - \Pi_2 (J^n + \frac{1}{2} \times \hat{n}) \right) \cdot \phi_h \times \hat{n} ds. \] (3.30)

Similarly, integrating (2.10) with \( \psi = \psi_h \) from \( t = t_{n+\frac{1}{2}} \) to \( t = t_{n+\frac{3}{2}} \), then dividing by \( \tau \), and using the result to subtract (3.2), we can obtain the error equation for \( H \):

\[ \mu_0(\delta_t H^n + \frac{1}{\tau}, \psi_h) + (\nabla \times E^n_{h\eta}, \psi_h) = \mu_0(\delta_t H^n + \frac{1}{\tau}, \psi_h) + (\nabla \times E^n_{h\xi}, \psi_h) \]

\[ + \left( \frac{1}{\tau} \int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}} \nabla \times E dt - \nabla \times E^n + \frac{1}{2}, \psi_h \right) + \left( \frac{1}{\tau} \int_{t_n}^{t_{n+\frac{1}{2}}} K_s dt - K^n + \frac{1}{2}, \psi_h \right). \] (3.31)

Finally, integrating (2.11) with \( \psi = \psi_h \) from \( t = t_{n+\frac{1}{2}} \) to \( t = t_{n+\frac{3}{2}} \), then dividing by \( \tau \), and using the result to subtract (3.2), we can obtain the error equation for \( J \):

\[ \tau_0(\delta_t J^n_{h\eta}, \chi_h)_\Gamma + (J^n_{h\eta}, \chi_h)_\Gamma - \sigma_0 (E^n_{h\eta}, \chi_h)_\Gamma \]

\[ = \tau_0(\delta_t J^n_{h\xi}, \chi_h)_\Gamma + (J^n_{h\xi}, \chi_h)_\Gamma - \sigma_0 (E^n_{h\xi}, \chi_h)_\Gamma \]

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\[ + \left( \frac{1}{\tau} \int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}} \mathbf{J} \, dt - \mathbf{J}^{n+1}, \mathbf{X}_h \right)_\Gamma - \sigma_0 \left( \frac{1}{\tau} \int_{t_{n+\frac{1}{2}}}^{t_{n+\frac{3}{2}}} \mathbf{E} \, dt - \mathbf{E}^{n+1}, \mathbf{X}_h \right)_\Gamma. \] (3.32)

With the above error equations, we can prove the following error estimate for our scheme (3.1)-(3.3).

**Theorem 3.2.** For the scheme (3.1)-(3.3) with initial approximations (3.7)-(3.8), under the time step constraint (3.12) and the following regularity assumptions:

\[ E \in L^\infty(0, T; H^r(\text{curl}; \Omega)), \quad H \in L^\infty(0, T; H^r(\Omega)), \quad J \in L^\infty(0, T; L^2(\Gamma)), \] (3.33)

\[ \partial_t \mathbf{E} \in L^2(0, T; H^r(\text{curl}; \Omega)), \quad \partial_t \mathbf{K}_s, \partial_t (\nabla \times \mathbf{E}) \in L^2(0, T; L^2(\Omega)), \] (3.34)

\[ \partial_t (\nabla \times \mathbf{H}) \in L^2(0, T; (L^2(\Omega))^5), \quad \partial_t \mathbf{J}, \partial_t \mathbf{E} \in L^2(0, T; L^2(\Gamma)), \] (3.35)

we have: For any \( 0 \leq m \leq N_t - 1, \)

\[ \epsilon_0 ||\mathbf{E}_h^m - \mathbf{E}_h^m||^2 + \mu_0 ||H_{h_t}^{m+\frac{1}{2}} - H_{h_t}^{m+\frac{1}{2}}||^2 + \frac{\tau_0}{\sigma_0} ||J_{h_t}^{m+\frac{1}{2}} - J_{h_t}^{m+\frac{1}{2}}||^2 \leq C(\tau^3 + h^{2r-1}), \]

where the constant \( C > 0 \) is independent of \( \tau \) and \( h \), and \( r \geq 1 \) is the order of the basis functions in spaces \( U_h \) and \( V_h \).

**Proof.** Choosing \( \phi_h = 2\tau \mathbf{E}_{h|}\frac{n+1}{2}, \psi_h = 2\tau \mathbf{H}_{h|}\frac{n+1}{2}, \chi_h = \frac{2\tau}{\sigma_0} \mathbf{J}_{h|}\frac{n+1}{2} \) in (3.28)- (3.32), respectively, then adding them together, we have

\[ \epsilon_0 (||\mathbf{E}_{h|}\frac{n+1}{2}||^2 - ||\mathbf{E}_{h|}\frac{n}{2}||^2) + \mu_0 (||H_{h_t}^{n+\frac{1}{2}}||^2 - ||H_{h_t}^{n+\frac{1}{2}}||^2) + \frac{\tau_0}{\sigma_0} (||J_{h_t}^{n+\frac{1}{2}}||^2 - ||J_{h_t}^{n+\frac{1}{2}}||^2) \]

\[ + \frac{2\tau}{\sigma_0} ||\mathbf{J}_{h|}\frac{n+1}{2}||^2 \leq \tau \left[ (H_{h_t}^{n+\frac{1}{2}}, \nabla \times \mathbf{E}_{h|}\frac{n}{2}) - (H_{h_t}^{n+\frac{1}{2}}, \nabla \times \mathbf{E}_{h|}\frac{n+1}{2}) \right] \]

\[ + \tau \left[ (\mathbf{E}_{h|}\frac{n+1}{2}, \mathbf{J}_{h|}\frac{n+1}{2}) - (\mathbf{E}_{h|}\frac{n}{2}, \mathbf{J}_{h|}\frac{n+1}{2}) \right] \]

\[ + 2\tau \epsilon_0 (\delta_t H_{h|}\frac{\xi}{2}, \mathbf{E}_{h|}\frac{n+1}{2}) - 2\tau (H_{h|}\frac{n+1}{2}, \nabla \times \mathbf{E}_{h|}\frac{n+1}{2}) + 2\tau (\mathbf{J}_{h|}\frac{n+1}{2}, \mathbf{E}_{h|}\frac{n+1}{2}) \]

\[ + 2\tau (H_{h|}\frac{n+1}{2} - \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \mathbf{H} \, dt, \nabla \times \mathbf{E}_{h|}\frac{n+1}{2}) + 2\tau (\frac{1}{\tau} \int_{t_n}^{t_{n+1}} \mathbf{J} \, dt - \mathbf{J}_{h|}\frac{n+1}{2}, \mathbf{E}_{h|}\frac{n+1}{2}) \]

\[ + 2\tau \mu_0 (\delta_t H_{h|}\frac{n+1}{2}, \mathbf{H}_{h|}\frac{n+1}{2}) + 2\tau (\nabla \times \mathbf{E}_{h|}\frac{n+1}{2}, \mathbf{H}_{h|}\frac{n+1}{2}) \]

\[ + 2\tau (\frac{1}{\tau} \int_{t_n}^{t_{n+\frac{1}{2}}} \mathbf{H} \, dt - \mathbf{H}_{h|}\frac{n+1}{2}, \mathbf{H}_{h|}\frac{n+1}{2}) + 2\tau (\frac{1}{\tau} \int_{t_n}^{t_{n+\frac{1}{2}}} \mathbf{K} \, dt - \mathbf{K}_{h|}\frac{n+1}{2}, \mathbf{H}_{h|}\frac{n+1}{2}) \]

\[ + \frac{2\tau}{\sigma_0} \epsilon_0 (\delta_t \mathbf{J}_{h|}\frac{n+1}{2}, \mathbf{J}_{h|}\frac{n+1}{2}) - \frac{2\tau}{\sigma_0} (\mathbf{J}_{h|}\frac{n+1}{2}, \mathbf{J}_{h|}\frac{n+1}{2}) + 2\tau (\mathbf{E}_{h|}\frac{n+1}{2}, \mathbf{J}_{h|}\frac{n+1}{2}) \]

\[ + \frac{2\tau}{\sigma_0} (\frac{1}{\tau} \int_{t_n}^{t_{n+\frac{1}{2}}} \mathbf{J} \, dt - \mathbf{J}_{h|}\frac{n+1}{2}, \mathbf{J}_{h|}\frac{n+1}{2}) - 2\tau (\frac{1}{\tau} \int_{t_n}^{t_{n+\frac{1}{2}}} \mathbf{E} \, dt - \mathbf{E}_{h|}\frac{n+1}{2}, \mathbf{J}_{h|}\frac{n+1}{2}) \Gamma. \] (3.36)

Summing up (3.36) from \( n = 0 \) to any \( m \leq N_t - 2 \), we obtain

\[ \epsilon_0 (||\mathbf{E}_{h|}\frac{n+1}{2}||^2 - ||\mathbf{E}_{h|}\frac{n}{2}||^2) + \mu_0 (||H_{h_t}^{n+\frac{1}{2}}||^2 - ||H_{h_t}^{n+\frac{1}{2}}||^2) + \frac{\tau_0}{\sigma_0} (||J_{h_t}^{n+\frac{1}{2}}||^2 - ||J_{h_t}^{n+\frac{1}{2}}||^2) \]

\[ + \frac{2\tau}{\sigma_0} \sum_{n=0}^{m} ||\mathbf{J}_{h|}\frac{n+1}{2}||^2 \leq \sum_{i=1}^{16} E r_i. \] (3.37)
Now we just need to estimate each $Err_i$. Similar to the proofs of (3.16) and (3.17), we immediately have

$$Err_1 = \tau \left[ (H_{h\eta}^\frac{1}{2}, \nabla \times E_{h\eta}^0) - (H_{h\eta}^{m+\frac{3}{2}}, \nabla \times E_{h\eta}^{m+1}) \right]$$

$$\leq \frac{1}{2} \tau C_v C_{in} h^{-1} (\mu_0 ||H_{h\eta}^\frac{1}{2}||^2 + \sigma_0 ||E_{h\eta}^0||^2) + \frac{1}{2} \tau C_v C_{in} h^{-1} (\mu_0 ||H_{h\eta}^{m+\frac{3}{2}}||^2 + \sigma_0 ||E_{h\eta}^{m+1}||^2), \quad (3.38)$$

and

$$Err_2 = \tau \left[ (E_{h\eta}^{m+1}, J_{h\eta}^{n+\frac{3}{2}})_{\Gamma} - (E_{h\eta}^0, J_{h\eta}^{\frac{1}{2}})_{\Gamma} \right]$$

$$\leq \frac{1}{2} \tau C_{tr} h^{-\frac{5}{2}} \sqrt{\frac{\sigma_0}{\sigma_0 \rho_0}} (\epsilon_0 ||E_{h\eta}^{m+1}||^2 + \frac{\tau_0}{\epsilon_0 \rho_0} ||J_{h\eta}^{m+\frac{3}{2}}||^2_{\Gamma}^2) + \frac{1}{2} \tau C_{tr} h^{-\frac{5}{2}} \sqrt{\frac{\sigma_0}{\sigma_0 \rho_0}} (\epsilon_0 ||E_{h\eta}^0||^2 + \frac{\tau_0}{\epsilon_0 \rho_0} ||J_{h\eta}^{\frac{1}{2}}||^2_{\Gamma}^2). \quad (3.39)$$

Using the inequality \((a, b) \leq \delta ||a||^2 + \frac{1}{\delta ||b||^2},\) Lemma 3.1 (i), and the interpolation error estimate (3.4), we have

$$Err_3 = \sum_{n=0}^{m} 2\tau \epsilon_0 (\delta \epsilon, E_{h\xi}^{n+\frac{3}{2}}, E_{h\eta}^{m+\frac{1}{2}}) \leq \sum_{n=0}^{m} 2\tau \epsilon_0 \left( \delta \epsilon ||E_{h\eta}^{m+\frac{1}{2}}||^2 + \frac{1}{4\delta^2} ||\delta \epsilon E_{h\xi}^{n+\frac{3}{2}}||^2 \right) \leq \tau \epsilon_0 \delta \sum_{n=0}^{m} (||E_{h\eta}^{n+1}||^2 + ||E_{h\eta}^n||^2) + \frac{\epsilon_0}{2\delta^2} \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} C h^{-2} ||\partial_t E_{\Gamma}||^2_{H^2(\text{curl}; \Omega)} dt. \quad (3.40)$$

Using the fact that $\nabla \times E_{h\eta}^{n+\frac{1}{2}} \in U_h$ and the projection operator property, we have

$$Err_4 = -2\tau \sum_{n=0}^{m} (H_{h\xi}^{n+\frac{1}{2}}, \nabla \times E_{h\eta}^{n+\frac{1}{2}}) = 0. \quad (3.41)$$

By the definition of (3.30), we have

$$Err_5 = 2\tau \sum_{n=0}^{m} (J_{h\eta}^{n+\frac{1}{2}}, E_{h\eta}^{n+\frac{1}{2}})_{\Gamma} = 0. \quad (3.42)$$

Using integration by parts, the PEC boundary condition (2.7), the inequality \((a, b) \leq \delta ||a||^2 + \frac{1}{\delta ||b||^2},\) and Lemma 3.1 (iii), we obtain

$$Err_6 = 2\tau \sum_{n=0}^{m} \left( \int_{t_n}^{t_{n+1}} \nabla \times H^{n+\frac{1}{2}} - \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \nabla \times E_{h\eta}^{n+\frac{1}{2}} \right) \leq 2\tau C_v \sum_{n=0}^{m} \left( \delta \epsilon_0 ||E_{h\eta}^{n+\frac{1}{2}}||^2 + \frac{\mu \delta}{4\delta^2} ||\nabla \times H^{n+\frac{1}{2}} - \frac{1}{\tau} \int_{t_n}^{t_{n+1}} \nabla \times H ||^2 \right) \leq \tau C_v \delta \epsilon_0 \sum_{n=0}^{m} (||E_{h\eta}^{n+1}||^2 + ||E_{h\eta}^n||^2) + \frac{\tau C_v \mu \delta}{8\delta^2} \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} ||\partial_t \nabla \times H||^2 dt. \quad (3.43)$$

By the trace inequality and Lemma 3.1 (iii), we have

$$Err_7 = 2\tau \sum_{n=0}^{m} \left( \frac{1}{\tau} \int_{t_n}^{t_{n+1}} J \ dt - J^{n+\frac{1}{2}}, E_{h\eta}^{n+\frac{1}{2}} \right)_{\Gamma} \leq 2\tau \sum_{n=0}^{m} \left( \frac{1}{\tau} \int_{t_n}^{t_{n+1}} J \ dt - J^{n+\frac{1}{2}} \right)_{\Gamma} \leq \frac{\tau}{\tau} \int_{t_n}^{t_{n+1}} J \ dt - J^{n+\frac{1}{2}} ||_{\Gamma} \cdot C_{tr} h^{-\frac{5}{2}} ||E_{h\eta}^{n+\frac{1}{2}}||.$$
By the trace inequality and the interpolation error estimate (3.4), we have

\[ \tau_\delta \sum_{n=0}^{m} \left( \tau_0 \sum_{n=0}^{m} \left( ||E_{h_\eta}^{n+\frac{1}{2}}||^2 + ||E_{h_\eta}^{n+\frac{1}{2}}||^2 \right) + \frac{C_{\tau_\delta}^2}{8 \delta \tau_0} \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} ||\partial_t J||_V^2 \, dt \right) \leq \tau_\delta \sum_{n=0}^{m} \left( ||E_{h_\eta}^{n+\frac{1}{2}}||^2 + ||E_{h_\eta}^{n+\frac{1}{2}}||^2 \right) + \frac{C_{\tau_\delta}^2}{8 \delta \tau_0} \sum_{n=0}^{m} \int_{t_n}^{t_{n+1}} ||\partial_t J||_V^2 \, dt. \] (3.44)

By the \(L^2\) projection property, we have

\[ Err_8 = 2 \tau \mu_0 \sum_{n=0}^{m} (\delta_t H_{h_\xi}^{n+1}, \overline{H}_{h_\eta}^{n+1}) = 0. \] (3.45)

Using the interpolation error estimate (3.4), we have

\[ Err_9 = 2 \tau \sum_{n=0}^{m} (\nabla \times E_{h_\xi}^{n+1}, \overline{H}_{h_\eta}^{n+1}) \leq 2 \tau C_v \sum_{n=0}^{m} \left( \frac{\delta_9 \mu_0}{4 \delta_9} ||\nabla \times E_{h_\xi}^{n+1}||^2 \right) \leq \tau C_v \delta_9 \mu_0 \sum_{n=0}^{m} \left( ||H_{h_\eta}^{n+\frac{1}{2}}||^2 + ||H_{h_\eta}^{n+\frac{1}{2}}||^2 \right) + \frac{\tau C_v \mu_0}{4 \delta_9} \sum_{n=0}^{m} \sum_{n=0}^{m} ||\nabla \times E_{h_\xi}^{n+1}||^2 \right). \] (3.46)

By Lemma 3.1 (iii), we have

\[ Err_{10} = 2 \tau \sum_{n=0}^{m} \left( \frac{1}{\tau} \int_{t_n+\frac{1}{2}}^{t_{n+1}} \nabla \times E \, dt - \nabla \times E_{h_\xi}^{n+1}, \overline{H}_{h_\eta}^{n+1} \right) \leq 2 \tau C_v \sum_{n=0}^{m} \left( \frac{\delta_9 \mu_0}{4 \delta_9} ||\nabla \times E_{h_\xi}^{n+1}||^2 \right) \leq \tau C_v \delta_9 \mu_0 \sum_{n=0}^{m} \left( ||H_{h_\eta}^{n+\frac{1}{2}}||^2 + ||H_{h_\eta}^{n+\frac{1}{2}}||^2 \right) + \frac{\tau C_v \mu_0}{4 \delta_9} \sum_{n=0}^{m} \sum_{n=0}^{m} ||\nabla \times E_{h_\xi}^{n+1}||^2 \right). \] (3.47)

Similar to \(Err_{10}\), we have

\[ Err_{11} = 2 \tau \sum_{n=0}^{m} \left( \frac{1}{\tau} \int_{t_n+\frac{1}{2}}^{t_{n+1}} K_{\xi} \, dt - K_{h_\xi}^{n+1}, \overline{H}_{h_\eta}^{n+1} \right) \leq \tau C_v \delta_{11} \mu_0 \sum_{n=0}^{m} \left( ||H_{h_\eta}^{n+\frac{1}{2}}||^2 + ||H_{h_\eta}^{n+\frac{1}{2}}||^2 \right) + \frac{\tau C_v \mu_0}{8 \delta_{11}} \sum_{n=0}^{m} \sum_{n=0}^{m} ||\partial_t K_{\xi}||^2 \right) dt. \] (3.48)

By the definition of (3.30), we obtain

\[ Err_{12} = \frac{2 \tau \tau_0}{\sigma_0} \sum_{n=0}^{m} \langle \delta_t J_{h_\xi}^{n+1}, \overline{J}_{h_\eta}^{n+1} \rangle_\Gamma = 0, \] (3.49)

and

\[ Err_{13} = \frac{2 \tau}{\sigma_0} \sum_{n=0}^{m} \langle J_{h_\xi}^{n+1}, \overline{J}_{h_\eta}^{n+1} \rangle_\Gamma = 0. \] (3.50)

By the trace inequality and the interpolation error estimate (3.4), we have

\[ Err_{14} = -2 \tau \sum_{n=0}^{m} \langle E_{h_\xi}^{n+1}, \overline{J}_{h_\eta}^{n+1} \rangle_\Gamma \leq 2 \tau \sum_{n=0}^{m} C_{\tau} \mu_0 \sum_{n=0}^{m} ||\nabla \times E_{h_\xi}^{n+1}|| \cdot ||\overline{J}_{h_\eta}^{n+1}||_\Gamma. \]
\[
\leq \tau \delta_{14} \sum_{n=0}^{m} \left( \|J_{h_{n+1}}^n\|_2^2 + \|J_{h_{n+1}}^n\|_1^2 \right) + \frac{\tau C_{2r}^2 h^{2r-1}}{2 \delta_{14}} \sum_{n=0}^{m} \|E\|_{L^\infty(0,T;H^r(\text{curl},\Omega))}. \quad (3.51)
\]

By Lemma 3.1 (ii), we have
\[
\text{Err}_{15} = \frac{2\tau}{\sigma_0} \sum_{n=0}^{m} \frac{1}{\sigma_0} \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \left( J - J_{h_{n}}^{n+1} \right) \text{dt} \]
\[
\leq \frac{\tau \delta_{15}}{\sigma_0} \sum_{n=0}^{m} \left( \|J_{h_{n+1}}^n\|_2^2 + \|J_{h_{n+1}}^n\|_1^2 \right) + \frac{\tau^4}{8 \delta_{15} \sigma_0} \sum_{n=0}^{m} \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \|\partial_t J\|_1^2 \text{dt}. \quad (3.52)
\]

Similarly, by Lemma 3.1 (iii), we have
\[
\text{Err}_{16} = -2\tau \sum_{n=0}^{m} \frac{1}{\sigma_0} \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} E \text{dt} - E_{n+1}^{m+1}, \frac{J_{h_{n}}^{n+1}}{\Gamma} \]
\[
\leq \tau \delta_{16} \sum_{n=0}^{m} \left( \|J_{h_{n+1}}^n\|_2^2 + \|J_{h_{n+1}}^n\|_1^2 \right) + \frac{\tau^4}{8 \delta_{16} \sigma_0} \sum_{n=0}^{m} \int_{t_{n+\frac{1}{2}}}^{t_{n+1}} \|\partial_t E\|_1^2 \text{dt}. \quad (3.53)
\]

Substituting the above estimates of \text{Err}_{i} into (3.37), combining like terms together, and dropping the last nonnegative term on the left hand side, we obtain
\[
\epsilon_0 \left( \|E_{h_{n+1}}^{m+1}\| - \|E_{h_{n}}^0\|^2 \right) + \mu_0 \left( \|H_{h_{n+1}}^{m+1}\|^2 - \|H_{h_{n}}^{m+1}\|^2 \right) + \frac{70}{\sigma_0} \left( \|J_{h_{n+1}}^{m+1}\|^2 - \|J_{h_{n}}^{m+1}\|^2 \right)
\]
\[
\leq \frac{1}{2} \tau C_v C_i h^{-1} \left( \mu_0 \|H_{h_{n}}^{m+1}\|^2 + \epsilon_0 \|E_{h_{n}}^0\|^2 \right) + \frac{1}{2} \tau C_v C_i h^{-1} \left( \|E_{h_{n}}^0\|^2 + \frac{\sigma_0}{\epsilon_0} \epsilon_0 \|E_{h_{n}}^0\|^2 + \frac{\tau_0}{\sigma_0} \|J_{h_{n}}^{m+1}\|_1^2 \right)
\]
\[
+ \left( \frac{1}{2} \tau C_v C_i h^{-1} + \tau C_v \delta_0 + \tau C_v \delta_{10} + \tau C_v \delta_{11} \right) \mu_0 \|H_{h_{n+1}}^{m+1}\|^2
\]
\[
+ \left( \frac{1}{2} \tau C_v C_i h^{-1} + \frac{1}{2} \tau C_v h^{-\frac{1}{2}} \left( \frac{\sigma_0}{\epsilon_0} \epsilon_0 + \tau \delta_3 + \tau C_v \delta_5 + \tau \delta_7 \right) \right) \epsilon_0 \|E_{h_{n}}^m+1\|^2
\]
\[
+ \left( \frac{1}{2} \tau C_v h^{-\frac{1}{2}} \left( \frac{\sigma_0}{\epsilon_0} \epsilon_0 + \tau \delta_4 \frac{\sigma_0}{\tau_0} + \tau \delta_5 \frac{\sigma_0}{\tau_0} + \tau \delta_6 \frac{\sigma_0}{\tau_0} \right) \right) \frac{70}{\sigma_0} \|J_{h_{n}}^{m+1}\|_1^2
\]
\[
+ \tau (2\delta_3 + 2C_v \delta_5 + 2\delta_T) \epsilon_0 \sum_{n=0}^{m} \|E_{h_{n}}^m\|^2 + \tau (2C_v \delta_9 + 2C_v \delta_{10} + 2C_v \delta_{11}) \mu_0 \sum_{n=0}^{m} \|H_{h_{n}}^{m+1}\|^2
\]
\[
+ \tau \left( \frac{2\sigma_0 \delta_{14}}{\tau_0} + \frac{2\delta_5}{\tau_0} + \frac{2\sigma_0 \delta_{16}}{\tau_0} \right) \frac{70}{\sigma_0} \sum_{n=0}^{m} \|J_{h_{n}}^{m+1}\|_1^2\right)
\]
\[
+ \frac{\epsilon_0 C_v h^{2r}}{2 \delta_3} \int_0^T \|\partial_t E\|_{H^r(\text{curl})}^2 \text{dt} + \frac{TC_v C_i \sigma_0 h^{2r}}{2 \delta_3} \int_0^T \|E\|_{L^\infty(0,T;H^r(\text{curl},\Omega))}^2 \text{dt}
\]
\[
+ \frac{TC_v C_i h^{2r-1}}{2 \delta_4} \int_0^T \|\partial_t E\|_{L^\infty(0,T;H^r(\text{curl},\Omega))}^2 \text{dt}
\]
\[
+ \frac{C_v \mu_0}{8 \sigma_0} \int_0^T \|\partial_t \nabla \times H\|^2 \text{dt} + \left( \frac{C_v \mu_0}{8 \sigma_0} \right) \int_0^T \|\partial_t J\|_1^2 \text{dt}
\]
\[
+ \int_0^T \|\partial_t \nabla \times E\|^2 \text{dt} + \int_0^T \|\partial_t K\|_1^2 \text{dt} + \frac{r^4 C_v \sigma_0}{8 \delta_1} \int_0^T \|\partial_t E\|_1^2 \text{dt}. \quad (3.54)
\]

Under the same time step constraint (3.12), by using the discrete Gronwall inequality and choosing those \delta_i properly, such as
\[
\delta_9 = \delta_{10} = \delta_{11} = \frac{1}{8C_v}, \quad \delta_3 = \delta_7 = \frac{1}{16}, \quad \delta_6 = \frac{1}{8C_v}, \quad \delta_14 = \delta_{16} = \frac{70}{8\sigma_0}, \quad \delta_15 = \frac{70}{8},
\]

12
we have
\[ \epsilon_0 \| E_{h^2}^{n+1} \|^2 + \mu_0 \| H_{h^2}^{n+\frac{1}{2}} \|^2 + \frac{\tau_0}{\sigma_0} \| J_{h^2}^{n+\frac{1}{2}} \|^2 \leq C \left( \epsilon_0 \| E_{h^2}^0 \|^2 + \mu_0 \| H_{h^2}^{n\frac{1}{2}} \|^2 + \frac{\tau_0}{\sigma_0} \| J_{h^2}^{n\frac{1}{2}} \|^2 + h^{2r-1} + \tau^3 \right) \exp(12(m+1)\tau) \leq C(h^{2r-1} + \tau^3), \] (3.55)

where in the last step we used the following initial approximation error estimates
\[ \| E_{h^2}^0 - E^0 \| \leq C \epsilon h^r, \quad \| H_{h^2}^{n\frac{1}{2}} - H^{n\frac{1}{2}} \| \leq C(h^r + \tau^2), \quad \| J_{h^2}^{n\frac{1}{2}} - J^{n\frac{1}{2}} \| \leq C(h^r + \tau^2). \] (3.56)

Finally, using the triangle inequality, the interpolation error estimate (3.4), and the \( L^2 \) projection error estimate, from (3.55) we conclude the proof. \( \Box \)

4 Numerical results

In this section, we present several numerical examples to demonstrate the effectiveness of our graphene model in simulating the propagation of surface plasmon polaritons (SPPs) on graphene sheets. Our numerical tests are carried out by using FEniCS [24].

4.1 Test of convergence rates

The first example is developed to test the theoretical convergence rate of our numerical scheme by a manufactured exact solution:

\[
E(x, y, t) = \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \sin(2\pi x)\sin(2\pi y)sin(2\pi t) \\ \cos(2\pi x)\cos(2\pi y)\sin(2\pi t) \end{pmatrix},
\]

\[
J(x, y, t) = \begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \frac{2}{1+4\pi^2} \sin(2\pi y)\sin(2\pi t) - 2\pi \cos(2\pi t) + 2\pi \exp(-t)) \\ \frac{1}{1+4\pi^2} \cos(2\pi x)\cos(2\pi y)\sin(2\pi t) - 2\pi \cos(2\pi t) + 2\pi \exp(-t)) \end{pmatrix},
\]

\[
H_1(x, y, t) = \frac{1}{1+4\pi^2} \sin(2\pi x)\sin(2\pi y)\sin(2\pi t),
\]

\[
H_2(x, y, t) = \frac{1}{1+4\pi^2} \sin(2\pi x)\sin(2\pi y)(2\pi \cos(2\pi t) - 2\pi \exp(-t)),
\]

which satisfies the following graphene model equations:

\[
\epsilon_0 \partial_t E = \nabla \times H_1 - J + f_1, \quad \text{in } \Omega_1, \quad \text{(4.1)}
\]

\[
\mu_0 \partial_t H_1 = -\nabla \times E + f_2, \quad \text{in } \Omega_1, \quad \text{(4.2)}
\]

\[
\tau_0 \partial_t J + J = \sigma_0 E, \quad \text{on } \Gamma, \quad \text{(4.3)}
\]

\[
\epsilon_0 \partial_t E = \nabla \times H_2 - J + f_3, \quad \text{in } \Omega_2, \quad \text{(4.4)}
\]

\[
\mu_0 \partial_t H_2 = -\nabla \times E + f_4, \quad \text{in } \Omega_2. \quad \text{(4.5)}
\]

Here the added source terms \( f_1, f_2, f_3 \) and \( f_4 \) can be calculated from the given exact solution \( E, H_1, H_2 \) and \( J \).

For simplicity, we choose the physical domain \( \Omega = (0, 1)^2 \), which is split into two subdomains \( \Omega_1 = (0, 1) \times (0, 1) \) and \( \Omega_2 = (0, 1) \times (0, 0.5) \) with interface \( \Gamma = \{ y = 0.5, x \in [0, 1] \} \). We apply
our developed scheme (3.1)-(3.3) to solve (4.1)-(4.5) with physical parameters $\epsilon_0 = \mu_0 = \tau_0 = \sigma_0 = 1$.

First, we solve this example with a fixed small time step size $\tau = 1 \times 10^{-4}$ and various mesh sizes for $N_t = 1000$ time steps. The obtained $L^2$ errors are presented in Tables 1 and 2 for the RTN finite element spaces $U_h$ and $V_h$ with $r = 1, 2$, respectively. Our results show that the obtained $L^2$ errors are at least $O(h^{r-0.5})$ for $r = 1, 2$, respectively.

Table 1: The errors obtained for Example 1 with $N_t = 1000$, $\tau = 1 \times 10^{-4}$, $r = 1$.

<table>
<thead>
<tr>
<th>h</th>
<th>$|E - E_h|_{L^2(\Omega)}$</th>
<th>rate</th>
<th>$|H - H_h|_{L^2(\Omega)}$</th>
<th>rate</th>
<th>$|J - J_h|_\Gamma$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>$1.9581 \times 10^{-2}$</td>
<td></td>
<td>$5.0621 \times 10^{-4}$</td>
<td></td>
<td>$9.8589 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>1/8</td>
<td>$9.9814 \times 10^{-3}$</td>
<td>0.9721</td>
<td>$2.4718 \times 10^{-4}$</td>
<td>1.0341</td>
<td>$5.0621 \times 10^{-4}$</td>
<td>0.9721</td>
</tr>
<tr>
<td>1/16</td>
<td>$5.0220 \times 10^{-3}$</td>
<td>0.9909</td>
<td>$1.1794 \times 10^{-4}$</td>
<td>1.0674</td>
<td>$5.0255 \times 10^{-5}$</td>
<td>0.9909</td>
</tr>
<tr>
<td>1/32</td>
<td>$2.5152 \times 10^{-3}$</td>
<td>0.9975</td>
<td>$5.4204 \times 10^{-5}$</td>
<td>1.1216</td>
<td>$2.5285 \times 10^{-5}$</td>
<td>0.9975</td>
</tr>
<tr>
<td>1/64</td>
<td>$1.2581 \times 10^{-3}$</td>
<td>0.9993</td>
<td>$2.3716 \times 10^{-5}$</td>
<td>1.1925</td>
<td>$1.2663 \times 10^{-5}$</td>
<td>0.9993</td>
</tr>
<tr>
<td>1/128</td>
<td>$6.3045 \times 10^{-4}$</td>
<td>0.9968</td>
<td>$1.2374 \times 10^{-5}$</td>
<td>0.9385</td>
<td>$3.1693 \times 10^{-6}$</td>
<td>0.9990</td>
</tr>
</tbody>
</table>

Table 2: The errors obtained for Example 1 with $N_t = 1000$, $\tau = 1 \times 10^{-4}$, $r = 2$.

<table>
<thead>
<tr>
<th>h</th>
<th>$|E - E_h|_{L^2(\Omega)}$</th>
<th>rate</th>
<th>$|H - H_h|_{L^2(\Omega)}$</th>
<th>rate</th>
<th>$|J - J_h|_\Gamma$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>$4.4129 \times 10^{-3}$</td>
<td></td>
<td>$2.5281 \times 10^{-4}$</td>
<td></td>
<td>$2.2218 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>1/8</td>
<td>$1.0730 \times 10^{-3}$</td>
<td>2.0400</td>
<td>$1.2988 \times 10^{-4}$</td>
<td>0.9607</td>
<td>$5.4012 \times 10^{-6}$</td>
<td>2.0403</td>
</tr>
<tr>
<td>1/16</td>
<td>$2.6160 \times 10^{-4}$</td>
<td>2.0361</td>
<td>$6.3906 \times 10^{-5}$</td>
<td>1.0232</td>
<td>$1.3113 \times 10^{-6}$</td>
<td>2.0422</td>
</tr>
<tr>
<td>1/32</td>
<td>$6.8419 \times 10^{-5}$</td>
<td>1.9349</td>
<td>$2.1809 \times 10^{-5}$</td>
<td>1.5510</td>
<td>$3.2979 \times 10^{-7}$</td>
<td>1.9913</td>
</tr>
<tr>
<td>1/64</td>
<td>$2.1824 \times 10^{-5}$</td>
<td>1.6484</td>
<td>$4.0333 \times 10^{-6}$</td>
<td>2.4349</td>
<td>$9.2774 \times 10^{-8}$</td>
<td>1.8297</td>
</tr>
<tr>
<td>1/128</td>
<td>$6.9954 \times 10^{-6}$</td>
<td>1.6414</td>
<td>$1.0048 \times 10^{-6}$</td>
<td>2.0050</td>
<td>$2.9704 \times 10^{-8}$</td>
<td>1.6430</td>
</tr>
</tbody>
</table>

Then we test the convergence rate in terms of $\tau$ by fixing $\tau = \frac{h}{200}$ to guarantee the stability constraint. The obtained $L^2$ errors are presented in Tables 3-4 for $r = 1, 2$, respectively, and they are at least $O(\tau^{1.5})$. When $r = 1$, due to the time step constraint $\tau = O(h)$, the theoretical convergence rate should be dominated by $O(h^{0.5}) = O(\tau^{0.5})$, but our numerical errors are better and almost $O(h)$.

4.2 Simulation of surface plasmon polaritons along the graphene sheets

To simulate the SPP phenomenon on the graphene sheet, we need to use a PML to surround the physical domain $\Omega$. Here we adopt the 2D TEz Ziolkowski PML model in the PML region.
Table 3: The obtained errors obtained for $r = 1$ by fixing $\tau = \frac{h}{200}$.

<table>
<thead>
<tr>
<th>h</th>
<th>$|E - E_h|_{L^2(\Omega)}$</th>
<th>rate</th>
<th>$|H - H_h|_{L^2(\Omega)}$</th>
<th>rate</th>
<th>$|J - J_h|_\Gamma$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10</td>
<td>$8.0084 \times 10^{-3}$</td>
<td></td>
<td>$2.0027 \times 10^{-4}$</td>
<td></td>
<td>$4.1938 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>1/20</td>
<td>$4.0208 \times 10^{-3}$</td>
<td>0.9940</td>
<td>$9.3342 \times 10^{-5}$</td>
<td>1.1016</td>
<td>$2.0546 \times 10^{-5}$</td>
<td>0.9721</td>
</tr>
<tr>
<td>1/40</td>
<td>$2.0126 \times 10^{-3}$</td>
<td>0.9984</td>
<td>$4.1560 \times 10^{-5}$</td>
<td>1.1670</td>
<td>$1.0158 \times 10^{-5}$</td>
<td>0.9909</td>
</tr>
<tr>
<td>1/80</td>
<td>$1.0066 \times 10^{-3}$</td>
<td>0.9995</td>
<td>$1.8654 \times 10^{-5}$</td>
<td>1.1556</td>
<td>$5.0491 \times 10^{-6}$</td>
<td>0.9975</td>
</tr>
<tr>
<td>1/160</td>
<td>$5.0537 \times 10^{-4}$</td>
<td>0.9990</td>
<td>$1.0326 \times 10^{-5}$</td>
<td>0.8532</td>
<td>$2.5194 \times 10^{-6}$</td>
<td>0.9993</td>
</tr>
</tbody>
</table>

Table 4: The obtained errors obtained for $r = 2$ by fixing $\tau = \frac{h}{200}$.

<table>
<thead>
<tr>
<th>h</th>
<th>$|E - E_h|_{L^2(\Omega)}$</th>
<th>rate</th>
<th>$|H - H_h|_{L^2(\Omega)}$</th>
<th>rate</th>
<th>$|J - J_h|_\Gamma$</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/10</td>
<td>$6.7883 \times 10^{-4}$</td>
<td></td>
<td>$1.0984 \times 10^{-4}$</td>
<td></td>
<td>$3.5527 \times 10^{-6}$</td>
<td></td>
</tr>
<tr>
<td>1/20</td>
<td>$1.6779 \times 10^{-4}$</td>
<td>2.0164</td>
<td>$4.8473 \times 10^{-5}$</td>
<td>1.1801</td>
<td>$8.4889 \times 10^{-7}$</td>
<td>2.0652</td>
</tr>
<tr>
<td>1/40</td>
<td>$4.5943 \times 10^{-5}$</td>
<td>1.8687</td>
<td>$1.3049 \times 10^{-5}$</td>
<td>1.8931</td>
<td>$2.1569 \times 10^{-6}$</td>
<td>1.9765</td>
</tr>
<tr>
<td>1/80</td>
<td>$1.7540 \times 10^{-5}$</td>
<td>1.3892</td>
<td>$3.5181 \times 10^{-6}$</td>
<td>1.8911</td>
<td>$6.5890 \times 10^{-7}$</td>
<td>1.7108</td>
</tr>
<tr>
<td>1/160</td>
<td>$6.9715 \times 10^{-6}$</td>
<td>1.3311</td>
<td>$9.6548 \times 10^{-7}$</td>
<td>1.8654</td>
<td>$2.1517 \times 10^{-8}$</td>
<td>1.6145</td>
</tr>
</tbody>
</table>
where \(\sigma_x(x)\) and \(\sigma_y(y)\) are the nonnegative damping functions in the \(x\) and \(y\) directions, respectively, the diagonal matrices \(D_i\) \((i = 1, 2, 3)\) are given as follows:

\[
D_1 = \text{diag}(\sigma_y - \sigma_x, \sigma_x - \sigma_y), \quad D_2 = \text{diag}(\sigma_x, \sigma_y), \quad D_3 = \text{diag}(\sigma_x(\sigma_x - \sigma_y), \sigma_y(\sigma_y - \sigma_x)).
\]

We propose the following finite element scheme for the above PML model in \(\Omega_{pml}\): For any \(n \geq 0\), find \(E^{n+1}_h, J^{n+\frac{1}{2}}_h \in V_h, H^{n+\frac{1}{2}}_h, K^{n+1}_h \in U_h\) such that

\[
\epsilon_0 \partial_t E = -\epsilon_0 D_1 E + \nabla \times H_z - J, \quad \text{in} \quad \Omega_{pml},
\]

\[
\mu_0 \partial_t H = -\mu_0 (\sigma_x + \sigma_y) H_z - \nabla \times E - K_z, \quad \text{in} \quad \Omega_{pml},
\]

\[
\partial_t J = -D_2 J + \epsilon_0 D_3 E, \quad \text{in} \quad \Omega_{pml},
\]

\[
\partial_t K_z = \mu_0 \sigma_x \sigma_y H_z, \quad \text{in} \quad \Omega_{pml},
\]

where \(\sigma_x(x)\) and \(\sigma_y(y)\) are the nonnegative damping functions in the \(x\) and \(y\) directions, respectively, the diagonal matrices \(D_i\) \((i = 1, 2, 3)\) are given as follows:

\[
D_1 = \text{diag}(\sigma_y - \sigma_x, \sigma_x - \sigma_y), \quad D_2 = \text{diag}(\sigma_x, \sigma_y), \quad D_3 = \text{diag}(\sigma_x(\sigma_x - \sigma_y), \sigma_y(\sigma_y - \sigma_x)).
\]

To simplify the implementation, we merge the graphene scheme (3.1)-(3.3) and the PML scheme (4.11)-(4.14) together by using subdomain dependent coefficients and rewrite them as follows:

\[
(\epsilon_0 (I + \frac{\tau D_1}{2}) E^{n+1}_h, \phi_h) = (\epsilon_0 (I - \frac{\tau D_1}{2}) E^n_h, \phi_h) + \tau (H^{n+\frac{1}{2}}_h, \nabla \times \phi_h) + \tau (J^{n+\frac{1}{2}}_h, \phi_h) - \tau C_{id} J^{n+\frac{1}{2}}_h, \phi_h),
\]

\[
(\mu_0 (\sigma_x + \sigma_y) H^{n+\frac{1}{2}}_h, \psi_h) = (\mu_0 (1 - \frac{\tau (\sigma_x + \sigma_y)}{2}) H^n_h, \psi_h) - \tau (\nabla \times E^{n+1}_h, \psi_h) - \tau C_{id} K^{n+1}_h, \psi_h),
\]

\[
((I + \frac{\tau D_2}{2}) J^{n+\frac{1}{2}}_h, \chi_h) + ((1 + \frac{\tau}{2}) J^{n+\frac{1}{2}}_h, \chi_h) = (I - \frac{\tau D_2}{2}) J^n_h, \chi_h) + \tau (\epsilon_0 D_3 E^{n+1}_h, \psi_h) + \tau (C_{id} J^{n+\frac{1}{2}}_h, \chi_h) + \tau (\sigma_0 E^{n+1}_h, \chi_h) - \tau C_{id} K^{n+1}_h, \chi_h),
\]

where we denote the identity matrix \(I = \text{diag}(1,1)\), write \(H_{zh}\) and \(J_{zh}\) in the PML subdomain as \(H_h\) and \(J_h\), and use the subdomain identity function

\[
C_{id} = \begin{cases} 
0, & \text{if } x \in \Omega, \\
1, & \text{if } x \in \Omega_{pml}.
\end{cases}
\]
In our simulation, we choose a physical domain \( \Omega = [-30, 30] \mu m \times [-10, 10] \mu m \), which is surrounded by the Ziolkowski PML with thickness \( 12h_x \) and \( 12h_y \) in the \( x \) and \( y \) directions, respectively, where \( h_x \) and \( h_y \) are the mesh sizes in the \( x \) and \( y \) directions, respectively. We use a uniformly refined triangular mesh with \( 128 \times 128 \) rectangles bisected into triangles.

The damping functions \( \sigma_x \) and \( \sigma_y \) for the PML are chosen as a fourth order polynomial:

\[
\sigma_x(x) = \begin{cases} 
\sigma_{\text{max}}(\frac{|x|-30}{dd})^4, & \text{when } |x| \geq 30, \\
0, & \text{elsewhere},
\end{cases}
\]

where the coefficient \( \sigma_{\text{max}} = -\log(\text{err}) \cdot 5 \cdot C_v/(2 \cdot dd) \) with \( \text{err} = 10^{-7} \), and \( dd \) denotes the thickness of the PML in the \( x \) direction. The function \( \sigma_y \) has the same form but varies with respect to the \( y \) variables.

**Example 1: A straight graphene sheet**

In this example, we present a simulation of SPPs along one graphene sheet aligned horizontally. The simulation setup is shown in Fig. 1, where a graphene sheet of 40\( \mu m \) long is embedded in domain \( \Omega \). Outside of \( \Omega \) is surrounded by the PML.

![Figure 1: The setup demonstration (with a coarse mesh) for Example 1.](image)

A pair of dipole source waves are placed at points \((-27, 1) \mu m \) and \((-27, -1) \mu m \), and imposed as \( K_s = \sin(2\pi f_0 t)/h_y \) and \( K_s = -\sin(2\pi f_0 t)/h_y \), respectively. In our simulation, we choose frequency \( f_0 = 10 \text{ THz} \), relaxation time \( \tau_0 = 1.2 \text{ ps} \), and the surface conductivity \( \sigma_0 \) given by the formula:

\[
\sigma_0 = \frac{-\tilde{q}^2 k_B T \tau_0}{\pi \hbar^2} \left( \frac{\mu_c}{k_B T} + 2 \ln(\exp(-\frac{\mu_c}{k_B T}) + 1) \right),
\]

where the electron charge \( q = 1.6022e - 19 \), the Kelvin temperature \( T = 300 \text{ K} \), the reduced Plank constant \( \hbar = 1.0546e - 34 \), the Boltzman constant \( k_B = 1.3806e - 23 \), and the chemical potential \( \mu_c = 1.5 \text{ eV} \).

We use the time step \( \tau = 8.3 \times 10^{-17} \text{ s} \), and run the simulation for 10000 time steps. Some snapshots of the obtained magnetic field \( H_z \) are shown in Fig. 2, which clearly show the SPPs propagate along the graphene sheet.
Example 2: Four adjacent graphene sheets

In this example, we simulate the wave propagation along four adjacent graphene sheets by our FETD scheme. The simulation setup is shown in Fig. 3, where four graphene sheets of length $10 \ \mu m$ each is embedded in domain $\Omega_0$. A pair of dipole incident waves are placed at points $(-27, 3.12) \ \mu m$ and $(-27, -3.12) \ \mu m$. We use the same simulation parameters as Example 1. Some snapshots of the magnetic field $H_z$ are presented in Fig. 4, which shows clearly that the SPPs propagate along the graphene sheets as demonstrated in the previous work [38].

Figure 2: Example 1. Contour plots of $H_z$ at 1000, 2000, 4000, 6000, 8000, and 10000 time steps.

Figure 3: Example 2. The setup (shown in a coarse mesh) for four adjacent graphene sheets buried in $\Omega$. 
Example 3: A tilted graphene sheet

This example is developed to simulate the propagation of SPPs along a tilted graphene sheet by our FETD scheme. The simulation setup is shown in Fig. 5, where one tilted graphene sheet situating on the line $y = \frac{1}{3}x$ with length $20\sqrt{5} \, \mu m$ is embedded in domain $\Omega_0$. A pair of dipole source waves are placed at points $(-21, -6) \, \mu m$ and $(-21, -8) \, \mu m$. The rest of the simulation data are the same as Example 1. The calculated magnetic fields $H_z$ obtained at different time steps are presented in Fig. 6, which shows that the SPPs also propagate along this tilted graphene sheet.

Figure 5: Example 3. The setup for the tilted graphene sheet.
Example 4: SPPs propagating along along a bifurcated graphene sheet

Finally, we present a bifurcated graphene sheet to demonstrate the flexibility of our FETD scheme to handle a complicated geometry. The simulation setup is illustrated in Fig. 7, and the rest simulation data are kept the same as Example 1. The obtained numerical magnetic fields $H_z$ at various time steps are presented in Fig. 8, which shows that the SPPs can propagate along this complicated graphene sheet.
5 Conclusion

In this paper, we develop a new formulation to simulate the surface plasmon polaritons propagating on graphene sheets. We treat the graphene as a thin sheet of current with an effective conductivity. A novel finite element method is proposed for solving this graphene model. Numerical results demonstrate the effectiveness of this graphene model for simulating the surface plasmon polaritons propagating on graphene sheets. The current error estimate is sub-optimal and caused by the graphene interface. We will continue exploring more efficient and optimally convergent schemes in the future.

6 Declarations

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Conflicts of Interests/Competing Interest The authors have no conflicts of interest to declare that are relevant to the content of this article.

References