ON THE SIMULATION OF INCOMPRESSIBLE, MISCIBLE DISPLACEMENT IN A NATURALLY FRACTURED PETROLEUM RESERVOIR

Todd Arbogast*

Communicated by J. Douglas

Résumé. — Une modèle ≪porosité double, permeabilité double≫ est presentée pour la simulation d'un déplacement incompressible miscible dans un mileau poreux naturellement fracturé. L'écoulement des fluides est décrit dans le système des fractures et dans chaque bloc matriciel par une équation elliptique pour la pression et une équation parabolique pour la concentration. L'échange de fluide entre les fractures et les blocs est modelé par l'imposition des conditions au bord sur les équations matricielles et par l'inclusion d'un terme de source distribuée en les équations dans les fractures. Une méthode d'éléments finis est donné pour approcher la solution. Elle utilise une méthode mixte pour la pression, une méthode modifiée des caractéristiques pour l'équation pour la concentration dans les fractures et une méthode ordinaire de Galerkin pour la concentration matricielle. La procédure converge asymtotiquement avec une vitesse optimale.

Abstract. — A double porosity/permeability model is presented to simulate an incompressible, miscible displacement in a naturally fractured petroleum reservoir. Fluid flow is described in the fracture system and in each matrix block by defining for each an elliptic pressure equation and a parabolic concentration equation. The matrix/fracture fluid transfer is modeled by imposing boundary conditions on the matrix equations and by including a macroscopically distributed source/sink in the fracture equations. A finite element procedure is defined to approximate the solution. It uses mixed methods for the pressure equations, a modified method of characteristics for the fracture concentration equation, and standard Galerkin methods for the matrix concentration equations. It is shown that the procedure converges asymptotically at the optimal rate.

Key words. — porous medium, double porosity, miscible displacement, fractured reservoir, finite element method.

AMS (MOS) subject classifications. — 65M25, 65N30, 76S05.

§1. Introduction

1.1. Opening remarks. It is fairly well understood how to model the flow of two completely miscible, incompressible fluids in a single porosity reservoir. An elliptic pressure equation and a parabolic concentration equation are obtained [8], [9], [19].

It is not so clear, however, how to model such flow in a double porosity/permeability reservoir. The usual approach is to assume that the exchange of fluids between

* Department of Mathematics, Purdue University, West Lafayette, Indiana 47907

the matrix and fracture systems is always in a quasi-steady state [5], [29]. A single function is defined to describe the matrix/fracture fluid transfer. The primary term of this function is proportional to the difference between the matrix and fracture system pressures (or flow potentials). The constant of proportionality, as well as the addition of smaller terms, is the subject of much debate [27].

An alternate approach is to model the matrix/fracture fluid transfer explicitly [1], [2], [4], [11]. This should give a physically more meaningful model, and no ad hoc terms or parameters need appear.

Fluid flow within each (single porosity) matrix block can be modeled in the usual way. One obtains pressure and concentration equations posed over each block. The flow in the fractures affects the flow in an individual matrix block only at its surface. This can be modeled as a boundary condition on the equations.

Fluid flow in the fractures needs to be viewed macroscopically. Provided that the fracture system is highly interconnected and the matrix blocks are relatively small (as compared to the size of the entire reservoir), the system of fractures may be thought of as a porous medium distinct from that of the matrix. Flow in the fracture system can then be considered to be of Darcy type and spread out over the entire reservoir. Again, the usual set of equations describes this flow, except that an additional source/sink term must appear. The fluid that flows out of (or into) the matrix blocks through their surfaces will appear in (or leave) the fracture system. In some way, this transfer must be modeled as a macroscopically distributed source/sink.

The flow of a single phase fluid of constant compressibility has been modeled in this way [1], [2], [11]. An incompressible (immiscible) waterflood has also been modeled [4], [11]. Here we shall model an incompressible, miscible displacement. The model will be related to an extension of the single phase, single component model described in [2].

1.2. A preview of the rest of the paper. Using the approach described above, we shall derive our model in Section 2.

In Section 3, we shall define a finite element procedure for approximating the solution of the differential model. The approximation of a single porosity reservoir has recently received a good deal of attention by several authors [8], [9], [10], [14], [16], [17], [18], [19], [20], [25], [26]. We will adapt two of the existing methods to our double porosity/permeability model in such a way that the resulting procedure is particularly well suited to solution on parallel computers.

We shall observe in Section 3 that the solution to the fracture system concentration equation has a tendency to develop relatively sharp fronts, though the solutions to the matrix concentration equations do not. That is, the formally parabolic fracture concentration equation is more nearly hyperbolic in practice. It should be discretized in a manner that is capable of resolving the fronts in the solution. The matrix concentration equations may be handled in a more standard fashion.

For the matrix equations we will consider the adaptation of a procedure of Douglas, Ewing, and Wheeler that uses a mixed finite element method for the elliptic pressure equations and a relatively standard Galerkin method for the parabolic concentration equations [9], [10]. Concentration is the quantity of physical interest. We will see that the concentration equations depend on the pressures only through the Darcy velocities, so it is appropriate to approximate these directly with mixed methods. This will give a better approximation of the velocities (which is critical) than a

more standard Galerkin approach [19] would yield.

The above approach could be used for the fracture equations as well. The resulting procedure has been shown to converge at the optimal rate [3], and it is somewhat simpler than that to be presented below. However, it would be unsatisfactory unless a very small time step were used.

For the fracture equations, we shall use a modification of the method of characteristics considered by Douglas, Ewing, Russell, and Wheeler [13], [17], [18], [25]. For technical reasons, we shall consider the model in a spatially periodic setting. We shall move the fracture concentration forward in time along the (approximate) characteristics of the hyperbolic part of the equation. Along the characteristics, the concentration should change very little; consequently, we should be able to take a reasonably large time step.

Because the velocities change less rapidly in time than the concentrations [8], [10], [14], we shall allow for the use of a longer time step in solving the pressure equations than is used in solving the concentration equations. When the fracture concentration front passes an individual matrix block, it may be desirable to reduce the time step used in solving the matrix equations. We will allow for this possibility in the case of the matrix concentration equations.

Because the matrix blocks themselves are small, the linear systems that arise in the finite element approximation procedure can be expected to be fairly small. Hence, it is appropriate to use direct solution methods on them. We shall incorporate a technique [10] that allows one to refactor each such matrix only once per pressure time step while retaining the order of accuracy obtained by refactoring once per matrix concentration time step.

In the final section, which is easily half of the entire paper, we shall prove that solutions to our approximation procedure converge asymptotically at the optimal rate to the solution of the differential model. We shall present the analysis in a relatively simple form that will not properly account for the smaller time step used in solving the matrix concentration equations. To control the matrix/fracture coupling, we will need to impose a mild but evidently artificial hypothesis.

As in most of the analyses of miscible displacement, we shall assume that the external sources/sinks are not concentrated at points (i.e., wells), but are instead smoothly distributed over the entire reservoir. If we allowed actual wells, the solution would be singular near them, so any approximation attempt and analysis would require special treatment [20], [26].

§2. The Incompressible, Miscible Displacement Model

2.1. The quantities of physical interest. Let $\Omega \subset \mathbb{R}^3$ be a naturally fractured reservoir with disjoint, two-connected matrix blocks $\Omega_i \subset \Omega$. It is important that the diameters of the matrix blocks be small compared to the diameter of the reservoir itself. Since the fractures are thin, we shall simply assume that $\cup_i \overline{\Omega}_i = \overline{\Omega}$. Let $J = (0, T\,],\ T > 0$, be the time interval of interest.

Within the reservoir, two incompressible, completely miscible fluids will flow in a single phase. Some of the fluid flows in the interconnected system of fractures, while the rest flows in the matrix blocks. The fluid and porous structure both of the fracture system and of the matrix must be characterized. The quantities associated to the (macroscopic) fracture system are defined on Ω, while the quantities associated to

the ith matrix block are defined on Ω_i. We shall often use a single symbol to denote a matrix quantity. It will be defined on $\Omega_m = \cup_i \Omega_i$, and the location x determines the block in question. When a physical quantity exists for both of the fracture and matrix systems, we shall use an upper case letter for the fracture quantity and the corresponding lower case letter for the matrix.

Let the fluid properties pressure, Darcy velocity, and concentration (of one of the two components) of the fracture system be denoted by $P(x,t)$, $U(x,t)$, and $C(x,t)$, respectively. The corresponding quantities for the matrix fluid are then $p(x,t)$, $u(x,t)$, and $c(x,t)$. Let μ and ρ denote the viscosity and density of the fluid mixture, respectively; these depend on the concentration C or c.

The porous system properties of the reservoir are the permeability tensor, porosity, and diffusion/dispersion tensor. Let $K(x)$, $\Phi(x)$, and $D(x,U)$, respectively, denote these quantities for the fracture system, and let $k_i(x)$, $\phi_i(x)$, and $d_i(x,u)$ denote the corresponding quantities for the ith matrix block (where the subscript i may be omitted). The diffusion/dispersion tensors are velocity dependent. Following [9] and [23],

$$D(x,U) = \Phi(x)\{d_{mol} I + |U|[D_{long} E(U) + D_{trans} E^{\perp}(U)]\}, \tag{2.1.1}$$

where the (j,k)-th entry of the tensor $E(U)$ is

$$E_{jk}(U) = \frac{U_j U_k}{|U|^2} \tag{2.1.2}$$

and $E^{\perp}(U) = I - E(U)$. A similar expression defines $d(x,u)$:

$$d_i(x,u) = \phi_i(x)\{d_{mol} I + |u|[d_{long,i} E(u) + d_{trans,i} E^{\perp}(u)]\}. \tag{2.1.3}$$

The matrix permeabilities are much smaller than the fracture permeability; hence, the following physical assumption is customarily made: the matrix blocks do not directly interact with each other, nor with the external sources/sinks; they interact only with the fracture system.

2.2. The flow in the fracture system. The flow in the fracture system is described by the usual pressure and concentration equations [8], [9], [19], except that a macroscopically distributed source term must appear to take into account the fluid transfer between the matrix and fracture systems. The pressure equation will be derived first.

Darcy's law states that

$$U = -A^{-1}(C)\big(\nabla P - \gamma(C)\big) \qquad \text{in } \Omega \times J, \tag{2.2.1}$$

where $A^{-1}(x,C) = K(x)/\mu(C)$ and $\gamma(x,C)$ is the product of $\rho(C)$, the gravitational constant, and the downward directed unit vector. We assume that the fluids do not change volume due to their mixing; that is, $\rho(C)$ is just the linear interpolant of the (constant) densities of the two components. Then incompressibility and the assumption that the external volumetric source/sink $f(x,t)$ acts only on the fracture system requires that

$$\nabla \cdot U = f \qquad \text{in } \Omega \times J. \tag{2.2.2}$$

We have omitted any effect of the matrix above since no net volumetric fluid transfer occurs between the matrix and fracture systems. A boundary condition is needed;

simply take the no flow Neumann condition:

$$U \cdot \nu = 0 \qquad \text{on } \partial\Omega \times J, \tag{2.2.3}$$

where $\nu(x)$ is the outer unit normal to $\partial\Omega$. Then, $\int_\Omega f(x)\,dx = 0$ is required for consistency. Note that (2.2.1)–(2.2.3) defines P only up to a constant.

The concentration equation reflects conservation of mass of the fluid components. Net component mass flow does occur between the matrix and fracture systems, so a matrix source term must appear in this equation. Let the operator $g_i(x; c, u)$ denote the volumetric source/sink from the ith matrix block of the fluid component whose concentration is c (equivalently, C). This function will be defined below in Subsection 2.4. The fracture concentration equation is then

$$\Phi C_t + \nabla \cdot \big(CU - D(U)\nabla C\big) = C_{inj} f_+ + C(f - f_+) + \sum_i g_i(c, u) \qquad \text{in } \Omega \times J,$$

for which (2.2.2) gives the nondivergence form as

$$\Phi C_t + U \cdot \nabla C - \nabla \cdot D(U)\nabla C = (C_{inj} - C) f_+ + \sum_i g_i(c, u) \qquad \text{in } \Omega \times J, \tag{2.2.4}$$

where the subscript t denotes partial differentiation in time, $C_{inj}(x, t)$ is the concentration of the injected fluid, and $f_+(x, t)$ is the positive (injection) part of f. Requiring no net component flow across $\partial\Omega$ gives the condition, with (2.2.3),

$$D(U)\nabla C \cdot \nu = 0 \qquad \text{on } \partial\Omega \times J. \tag{2.2.5}$$

Finally, the initial fracture concentration $C^0(x)$ must be given:

$$C = C^0 \qquad \text{on } \Omega \times \{0\}. \tag{2.2.6}$$

2.3. The flow in the matrix. We will now consider the flow of fluid in the ith matrix block. The fracture system affects the block only at its surface, so the equations describing interior flow are the usual ones, with no external source/sink terms.

The pressure equation is

$$u = -a^{-1}(c)\big(\nabla p - \gamma(c)\big) \qquad \text{in } \Omega_i \times J \tag{2.3.1}$$

and

$$\nabla \cdot u = 0 \qquad \text{in } \Omega_i \times J, \tag{2.3.2}$$

where $a^{-1}(x, c) = k(x)/\mu(c)$ (and $a_i^{-1}(x, c) = k_i(x)/\mu(c)$).

The concentration equation is

$$\phi c_t - \nabla \cdot q(c, u) = 0 \qquad \text{in } \Omega_i \times J, \tag{2.3.3}$$

where we have written the volumetric flux as

$$q(c, u) = d(u)\nabla c - cu. \tag{2.3.4}$$

Again, the initial concentration $c^0(x)$ must be given:

$$c = c^0 \qquad \text{on } \Omega_i \times \{0\}. \tag{2.3.5}$$

At the surface of the block, matrix fluid contacts fracture fluid; we must enforce continuity of pressure and of concentration there. Pressure is easy:

$$p = P \qquad \text{on } \partial\Omega_i \times J. \tag{2.3.6}$$

Continuity of concentration cannot be imposed so straightforwardly. The matrix fluid that leaves the block at a point $x \in \partial\Omega_i$ must be macroscopically spread out so that it enters the fracture system in some region about x. It is inconsistent to have only the fracture fluid at x influence the matrix flow; all the fracture fluid in the region about x must affect the matrix flow. A consistent way for this to occur, which is related to the procedure of [2], will be described in the next subsection. There we will define an operator Λ_i so that continuity of concentration can be expressed as

$$c = \Lambda_i(C) \qquad \text{on } \partial\Omega_i \times J. \tag{2.3.7}$$

The matrix concentration initial and boundary conditions need to be consistent; hence, the following relation should hold:

$$c^0 = \Lambda_i(C^0) \qquad \text{on } \Omega_i. \tag{2.3.8}$$

2.4. The matrix/fracture coupling. Let $\{\chi_i(x)\}$ be some partition of unity over Ω such that each χ_i is or is approximately the characteristic function of Ω_i. Specifically, we require that the support of $\chi_i \approx \Omega_i$, $\int_\Omega \chi_i(x)\,dx = |\Omega_i| =$ volume of Ω_i, and $0 \le \chi_i$. The function χ_i will define and weight the region of space over which the ith block and the fracture system influence each other.

Since the blocks are small and the fracture flow is faster than the matrix flow, let us assume that, at each fixed time, the variation in C over Ω_i is small. Following [1], [4], and [11], we might like to assume that this variation is so small as to be negligible. However, in that case, it is not hard to see that no transfer of fluid would occur due to convection. Our fluids are incompressible and mix without changing volume, so there is no net volumetric flow into or out of an individual matrix block. If the concentration over the surface of a block was constant in space, then the net convection of a mass component in through the block's surface would exactly equal the net convection out (though diffusion and dispersion could lead to some net component mass flow). Since viscous displacement is an important physical process in the recovery of oil by miscible techniques, we must assume some variation in the concentration over the surface of each block. As in [2], we will assume that the variation in the fracture flow is essentially linear.

To define the linear nature of the fracture flow near Ω_i, let $\{\lambda_{i,0}, \ldots, \lambda_{i,3}\}$ be an orthonormal basis of the linear functions with respect to the inner product given by integration against the weight χ_i. Now define the linear operator Λ_i mapping $L^1(\Omega)$ onto the set of linear functions by

$$\Lambda_i(\omega)(x) = \sum_{j=0}^{3} \left(\int_\Omega \omega(y)\lambda_{i,j}(y)\chi_i(y)\,dy \right) \lambda_{i,j}(x). \tag{2.4.1}$$

A macroscopically spread matrix source/sink that is consistent with this is defined

as

$$g_i\big(x; c(\,\cdot\,,t), u(\,\cdot\,,t)\big) = -\sum_{j=0}^{3}\Big(\int_{\Omega_i}\{\phi(y)c_t(y,t)\lambda_{i,j}(y) + q\big(y, c(y,t), u(y,t)\big)\cdot\nabla\lambda_{i,j}(y)\}\,dy\Big)\lambda_{i,j}(x)\chi_i(x). \tag{2.4.2}$$

That is, the fluid that flows through $\partial\Omega_i$ is

$$-q(c,u)\cdot\nu_i = -\big(d_i(u)\nabla c - cu\big)\cdot\nu_i$$

(where $\nu_i(x)$ is the outer unit normal to $\partial\Omega_i$), and this distribution, supported on $\partial\Omega_i$, agrees with the action of g_i up to the linear order of the test function used:

$$-\int_{\partial\Omega_i} q(c,u)\cdot\nu_i\Lambda_i(\omega)\,ds = \int_\Omega g_i(c,u)\omega\,dx, \qquad \omega\in C^\infty(\Omega), \tag{2.4.3}$$

by the divergence theorem, (2.3.3), and the orthonormality of $\{\lambda_{i,0},\dots,\lambda_{i,3}\}$. With $\omega\equiv 1$ above, we see that (2.4.2) gives us a matrix source/sink that conserves mass in a global sense; it also conserves mass in a local sense. Without loss of generality, we can suppose that $\lambda_{i,0}$ is a constant. Then it is easy to see that the $j=0$ term in (2.4.2) alone accounts for the global conservation of mass, while the other terms simply locally redistribute the matrix source/sink somewhat. Consistency has been maintained in the mass transfer between the two porous media because each matrix block affects and is affected by the fracture system as if the fracture system flow were linear in the vicinity of the block.

§3. The Finite Element Approximation Procedure

In this section, we first consider whether the concentration equations tend to develop fronts, as this strongly influences our choice of an approximation procedure. In the second subsection, we clarify the mathematical assumptions that are tacitly assumed throughout the rest of the paper and define some general notation. Once this is done, in the third subsection, we can write down a weak form of our model. This weak form is appropriate for finite element approximation, the notation for which is given in Subsection 3.4. We are then ready in Subsection 3.5 to define our approximation procedure. Finally, in the last subsection, we include a few words on implementing the procedure on a computer.

3.1. The expectation of concentration fronts. It is well known that miscible displacements in unfractured petroleum reservoir simulation are dominated by the convection of the fluids rather than by their diffusion/dispersion [14], [17]. This means that the formally parabolic concentration equation is almost hyperbolic, and so fronts can and do develop in the concentration that are relatively sharp when viewed on the scale of the entire oil field. The sharpness of the front increases with the hyperbolic velocity, which is the Darcy velocity divided by the porosity (roughly speaking, this is the actual microscopic velocity of the fluid), and this velocity must be scaled by the size of the reservoir.

Estimates of the fracture system permeability and porosity and of the diameter of the matrix blocks have appeared in the petroleum literature [21], [27], [28], [29]. Typically, the fracture system may have a permeability that is forty to several hundred

times larger and a porosity that is ten to a few hundred times smaller than that for an unfractured reservoir. Hence, the hyperbolic velocity in the fracture system may be several thousand times greater than in an unfractured reservoir. It is difficult to estimate the effect of the matrix on the front, but it does act in a regularizing manner [1]. It will smooth out the front to some significant degree, but let it suffice to say that the fracture concentration will develop relatively sharp concentration fronts.

The matrix has a hyperbolic velocity comparable to that for an unfractured reservoir. However, the diameter of a matrix block is several hundred times smaller than that of the reservoir itself; therefore, the flow in a matrix block will have a much reduced tendency to develop fronts. It is also true that the small size of the block allows us to discretize it fairly finely if we wish, so that in any case the matrix concentration can be approximated rather easily.

3.2. Some mathematical assumptions and notation. We shall implicitly assume that all of the quantities defined in the second section above are sufficiently smooth; in particular, assume the quantities A, Φ, D, γ, f, C_{inj}, C^0, and c^0, and, for each i, a_i, ϕ_i, d_i, and χ_i to be so. In addition, Φ and ϕ shall be bounded above and below by positive constants, and A and a shall be uniformly positive definite. The same will be true of D and d once we assume that the molecular diffusion coefficient d_{mol} is strictly positive. For convenience in the analysis to follow, let

$$0 < \Phi_* \le \Phi, \qquad 0 < \phi_* \le \phi, \qquad \text{and} \qquad D_* = d_{mol}.$$

We shall assume that Ω is a two-connected domain in $\mathbb{R}^3$ with a smooth boundary. Each Ω_i should be convex; otherwise the solution may have some singular behavior near the reentrant corners.

Under these assumptions, it is reasonable to expect that the solution is smooth and regular enough that the approximation procedures to be defined below can be analyzed as in the last two sections of the paper. We shall tacitly assume that this is the case.

Each of our assumptions is physically reasonable except for those made on the external source/sink term f (as mentioned in the introduction).

Let us now define some additional notation, most of which is relatively standard. With $*$ being blank, i, or m, $W_*^{\alpha,\beta}$ will denote the usual Sobolev space of α times differentiable functions in the Lebesgue space L_*^β over a domain Ω_*, and $\|\cdot\|_{\alpha,\beta,*}$ will denote its norm. We will simply write H_*^α for the Hilbert space $W_*^{\alpha,2}$ and $\|\cdot\|_{\alpha,*}$ for its norm. Denote the inner product on L_*^2 or $\left(L_*^2\right)^3$ by $(\cdot\,,\cdot)_*$. Let $H_0^1(\Omega_*)$ denote the closure in H_*^1 of $C_0^\infty(\Omega_*)$, the infinitely differentiable functions of compact support in Ω_*. For a Banach space X and a nonnegative integer α, let $W^{\alpha,\beta}(J';X)$ denote the space of maps $\varphi : J' \longrightarrow X$ with the norm

$$\|\varphi\|_{W^{\alpha,\beta}(J';X)} = \left(\int_{J'} \sum_{j=0}^{\alpha} \left\| \frac{\partial^j \varphi}{\partial t^j} \right\|_X^\beta dt \right)^{1/\beta},$$

where the right-hand side must be modified in the usual way if $\beta = \infty$, and where J'

will be omitted if $J' = J$. It will be convenient to define the following spaces:

$$H(div;\Omega_*) = \{\psi \in \big(L^2(\Omega_*)\big)^3 : \nabla\cdot\psi \in L^2(\Omega_*)\},$$
$$H_N(div;\Omega_*) = \{\psi \in H(div;\Omega_*) : \psi\cdot\nu_* = 0 \text{ on } \partial\Omega_*\},$$
$$L^2_C(\Omega_*) = L^2(\Omega_*)/\{\varphi : \varphi \equiv \text{constant}\},$$
$$H^1_L(\Omega_*) = H^1_0(\Omega_*) + \text{span}\{\lambda_{i,0},\dots,\lambda_{i,3}\}.$$

Finally, let Q and ϵ denote generic positive constants, where ϵ may be taken as small as we please.

3.3. A weak form of the differential system. Assuming that the differential model has a bounded solution, it will satisfy the weak form of the equations that follows. The weak form in ii) below is easily seen by noting that (2.3.1) can be rewritten with (2.2.1) as

$$a(c)u + \nabla(p - P) = \gamma(c) - \nabla P = \gamma(c) + A(C)U - \gamma(C).$$

i) The fracture pressure equation:
Find maps $\{U, P\} : J \longrightarrow H_N(div;\Omega)\times L^2_C(\Omega)$ such that

$$\big(A(C)U,\psi\big) - (\nabla\cdot\psi, P) = \big(\gamma(C),\psi\big), \qquad \psi \in H_N(div;\Omega), \tag{3.3.1}$$
$$(\nabla\cdot U,\varphi) = (f,\varphi), \qquad \varphi \in L^2_C(\Omega). \tag{3.3.2}$$

ii) The ith matrix block pressure equation:
Find maps $\{u, p\} : J \longrightarrow H(div;\Omega_i)\times L^2_i$ such that

$$\begin{aligned}\big(a(c)u,\psi\big)_i - (\nabla\cdot\psi,p)_i &= \big(\gamma(c),\psi\big)_i + \big([A(C)U-\gamma(C)],\psi\big)_i \\ &\quad - (\nabla\cdot\psi,P)_i, \qquad \psi \in H(div;\Omega_i),\end{aligned} \tag{3.3.3}$$
$$(\nabla\cdot u,\varphi)_i = 0, \qquad \varphi \in L^2_i. \tag{3.3.4}$$

iii) The fracture concentration equation:
Find a map $C : J \longrightarrow H^1$ such that

$$\begin{aligned}(\Phi C_t + U\cdot\nabla C,\omega) + \big(D(U)\nabla C,\nabla\omega\big) + (Cf_+,\omega) \\ = (C_{inj}f_+,\omega) + \sum_i \big(g_i(c,u),\omega\big), \qquad \omega \in H^1,\end{aligned} \tag{3.3.5}$$
$$C(x,0) = C^0(x), \qquad \text{for } x \in \Omega. \tag{3.3.6}$$

iv) The ith matrix block concentration equation:
Find a map $c : J \longrightarrow H^1_L(\Omega_i)$ such that

$$(\phi c_t,\omega)_i + \big(q(c,u),\nabla\omega\big)_i = 0, \qquad \omega \in H^1_0(\Omega_i), \tag{3.3.7}$$
$$c(x,t) = \Lambda_i\big(C(\,\cdot\,,t)\big)(x), \qquad \text{for } (x,t) \in \partial\Omega_i\times J, \tag{3.3.8}$$
$$c(x,0) = c^0(x), \qquad \text{for } x \in \Omega_i. \tag{3.3.9}$$

Note that the boundary conditions (2.2.5) on C and (2.3.6) on p have been imposed implicitly above. Recall that g_i, q, and Λ_i are defined by (2.4.2), (2.3.4), and (2.4.1), respectively.

3.4. Discretization of space and time. For H_p and H_c in $(0,1]$, let $\mathcal{T}_{H_p}(\Omega)$ and $\mathcal{T}_{H_c}(\Omega)$ be partitions of Ω into simplices or rectangles of diameters bounded by H_p and H_c, respectively. Each of these families of partitions must satisfy the nondegeneracy condition, namely, that there is some bound on the ratio of the diameter of each simplex or rectangle to the diameter of the largest ball inscribed within it. Furthermore, the second family $\mathcal{T}_{H_c}(\Omega)$ must be fully quasi-regular; that is, there is also some overall bound on the ratio of the diameter of the largest simplex or rectangle to that of the smallest for each partition. Analogously, for h_p and h_c in $(0,1]$, partition each Ω_i into $\mathcal{T}_{h_p}(\Omega_i)$ and $\mathcal{T}_{h_c}(\Omega_i)$. Of course, h_p and h_c could vary from block to block, but for notational convenience we will not consider this possibility.

Let each of the spaces $\bar{\mathcal{V}}_{H_p} \times \bar{\mathcal{W}}_{H_p}$ and $\mathcal{V}_i \times \mathcal{W}_i = \mathcal{V}_{h_p,i} \times \mathcal{W}_{h_p,i}$ (for each i) be that of Raviart-Thomas-Nedelec [22], [24], Brezzi-Douglas-Fortin-Marini [7], or Brezzi-Douglas-Durán-Fortin [6] associated to the partitions $\mathcal{T}_{H_p}(\Omega)$ and $\mathcal{T}_{h_p}(\Omega_i)$, respectively, of indexes such that the approximation properties (3.4.1)–(3.4.6) below hold. (Actually, any mixed finite element spaces having the properties described in Subsection 4.2 below can be used.) Let $\mathcal{V} = \mathcal{V}_{H_p} = \bar{\mathcal{V}}_{H_p} \cap H_N(div;\Omega)$ and $\mathcal{W} = \mathcal{W}_{H_p} = \bar{\mathcal{W}}_{H_p}/\{\varphi : \varphi \equiv \text{constant}\} \subset L^2_C(\Omega)$. Then, for any $(v,w) \in H_N(div;\Omega) \times L^2_C(\Omega)$,

$$\inf_{\psi\in\mathcal{V}} \|v-\psi\|_0 \le Q\|v\|_R H_p^R, \qquad 0 \le R \le R^*, \tag{3.4.1}$$

$$\inf_{\psi\in\mathcal{V}} \|\nabla\cdot(v-\psi)\|_0 \le Q\|\nabla\cdot v\|_R H_p^R, \qquad 0 \le R \le R^{**}, \tag{3.4.2}$$

$$\inf_{\varphi\in\mathcal{W}} \|w-\varphi\|_0 \le Q\|w\|_R H_p^R, \qquad 0 \le R \le R^{**}, \tag{3.4.3}$$

where $R^{**} = R^* \ge 1$ for the first two spaces [7], [22], [24] and $R^{**} = R^* - 1 \ge 1$ for the other space [6], and, with r^{**} defined analogously, for any $(v,w) \in H(div;\Omega_i) \times L^2(\Omega_i)$,

$$\inf_{\psi\in\mathcal{V}_i} \|v-\psi\|_{0,i} \le Q\|v\|_{r,i} h_p^r, \qquad 0 \le r \le r^*, \tag{3.4.4}$$

$$\inf_{\psi\in\mathcal{V}_i} \|\nabla\cdot(v-\psi)\|_{0,i} \le Q\|\nabla\cdot v\|_{r,i} h_p^r, \qquad 0 \le r \le r^{**}, \tag{3.4.5}$$

$$\inf_{\varphi\in\mathcal{W}_i} \|w-\varphi\|_{0,i} \le Q\|w\|_{r,i} h_p^r, \qquad 0 \le r \le r^{**}. \tag{3.4.6}$$

Note that the numbers R^* and r^* may be as small as one.

Let $\mathcal{M} = \mathcal{M}_{H_c} \subset H^1$ and, for each i, $\mathcal{M}_i^0 = \mathcal{M}_{h_c,i}^0 \subset H_0^1(\Omega_i)$ be standard finite element spaces associated to $\mathcal{T}_{H_c}(\Omega)$ and $\mathcal{T}_{h_c}(\Omega_i)$, respectively, such that

$$\inf_{\omega\in\mathcal{M}} \|\Theta-\omega\|_1 \le Q\|\Theta\|_S H_c^{S-1}, \qquad 1 \le S \le S^*, \tag{3.4.7}$$

$$\inf_{\omega\in\mathcal{M}_i^0} \|\theta-\omega\|_{1,i} \le Q\|\theta\|_{s,i} h_c^{s-1}, \qquad 1 \le s \le s^*, \tag{3.4.8}$$

for any $\Theta \in H^1$ and $\theta \in H_0^1(\Omega_i)$. Here, S^* and s^* must be at least two. It will be convenient to define the space

$$\mathcal{M}_i^L = \mathcal{M}_i^0 + \text{span}\{\lambda_{i,0},\dots,\lambda_{i,3}\} \subset H_L^1(\Omega_i).$$

Next, let us define some notation that will be used to discretize the time variable. Schematically, J will be partitioned as shown in Diagram 3.4.1. First, some $\Delta t_c > 0$ is chosen to represent the fracture concentration time step. Then for some positive

integer l^*, we can define the matrix concentration time step δt_c by

$$\Delta t_c = l^* \delta t_c.$$

As we will see later in the analysis, the first pressure time step should be chosen to be smaller than the rest. Hence, for two positive integers $m^{*,0}$ and m^*, the first and succeeding pressure time steps will be

$$\Delta t_p^1 = m^{*,0} \Delta t_c \qquad \text{and} \qquad \Delta t_p = m^* \Delta t_c,$$

respectively. Finally, let n^* be the number of pressure time steps in J; that is,

$$T = \Delta t_p^1 + (n^* - 1)\Delta t_p$$

(extend the end time T if necessary).

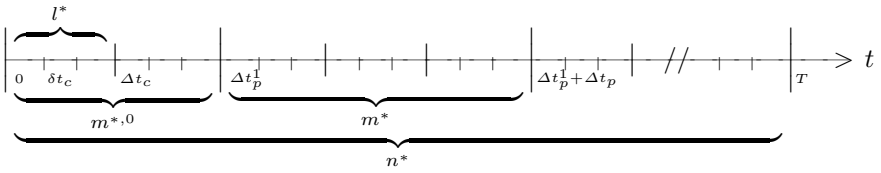

Diagram 3.4.1 — The time line.

For notational purposes, let $m^{*,n} = m^*$ for $n \geq 1$. In fact, there is no reason why Δt_p and even Δt_c and δt_c could not vary with time; moreover, while each block is near the fracture concentration front, a temporary reduction of the matrix discretization parameters is probably to be desired. This extension is straightforward but cumbersome, so we will not pursue it.

Let

$$t_{n,m,l} = \begin{cases} m\Delta t_c + l\delta t_c & \text{if } n = 0 \\ \Delta t_p^1 + (n-1)\Delta t_p + m\Delta t_c + l\delta t_c & \text{if } n \geq 1, \end{cases}$$

and

$$\varphi^{n,m,l} = \varphi(t_{n,m,l}),$$

where we omit l if $l = 0$ and both m and l if $m = l = 0$. Note that

$$\varphi^{n,m,l^*} = \varphi^{n,m+1}, \qquad \varphi^{n,m^{*,n}} = \varphi^{n+1}.$$

Also let

$$\partial\varphi^{n,m+1} = \frac{\varphi^{n,m+1} - \varphi^{n,m}}{\Delta t_c}, \qquad \delta\varphi^{n,m,l+1} = \frac{\varphi^{n,m,l+1} - \varphi^{n,m,l}}{\delta t_c},$$

and

$$J_{n+1} = (t_n, t_{n+1}], \qquad J_{n,m+1} = (t_{n,m}, t_{n,m+1}].$$

The sum of $\varphi^{n,m+1}$ over all n and m for which $0 < t_{n,m+1} \leq t_{N,M+1}$ shall be denoted simply by

$$\sum_{n,m}^{N,M} \varphi^{n,m+1} = \sum_{n=0}^{N-1} \sum_{m=0}^{m^{*,n}-1} \varphi^{n,m+1} + \sum_{m=0}^{M} \varphi^{N,m+1}.$$

We will need to extrapolate functions defined at pressure time levels. We can use a two point linear extrapolation beyond time t_1, and a constant extrapolation over the first pressure time interval. Hence let

$$E\varphi^{n,m,l} = \begin{cases} \varphi^0 & \text{if } n = 0 \\ \varphi^n + \frac{1}{m^{*,n-1}}\big(m + \frac{l}{l^*}\big)(\varphi^n - \varphi^{n-1}) & \text{if } n \geq 1, \end{cases}$$

where, again, we will drop the l if $l = 0$. We will also extrapolate to the midpoints of the pressure time intervals. So, for half integers, let

$$E\varphi^{n+\frac{1}{2}} = \begin{cases} \varphi^0 & \text{if } n = 0 \\ \varphi^n + \frac{m^{*,n}}{2m^{*,n-1}}(\varphi^n - \varphi^{n-1}) & \text{if } n \geq 1. \end{cases}$$

Finally, denote interpolation of the fracture concentration time levels by

$$I\omega^{n,m,l} = \left(1 - \frac{l}{l^*}\right)\omega^{n,m} + \frac{l}{l^*}\omega^{n,m+1}.$$

We will approximate U, P, and C by $U_h \in \mathcal{V}$, $P_h \in \mathcal{W}$, and $C_h \in \mathcal{M}$, respectively, and we will approximate u, p, and c on each block Ω_i by $u_h \in \mathcal{V}_i$, $p_h \in \mathcal{W}_i$, and $c_h \in \mathcal{M}_i^L$, respectively.

We will approximate the hyperbolic part of the fracture concentration equation by viewing it as a directional derivative in (x,t)-space [13]. Let

$$\tau(x,t) = \frac{\big(U(x,t), \Phi(x)\big)}{\sqrt{|U(x,t)|^2 + \Phi(x)^2}} \tag{3.4.9}$$

denote the unit vector in the characteristic direction (U, Φ). Then the hyperbolic part of the fracture concentration equation is

$$\Phi C_t + U \cdot \nabla C = \sqrt{|U|^2 + \Phi^2}\frac{\partial C}{\partial \tau}, \tag{3.4.10}$$

and this will be approximated at $(x, t_{n,m+1})$ by

(3.4.11)

$$\begin{aligned} \sqrt{|U^{n,m+1}|^2 + \Phi^2}\frac{\partial C^{n,m+1}}{\partial \tau} &\approx \Phi\frac{C^{n,m+1} - C^{n,m}\big(x - \frac{U^{n,m+1}}{\Phi}\Delta t_c\big)}{\Delta t_c} \\ &\approx \Phi\frac{C_h^{n,m+1} - \hat{C}_h^{n,m}}{\Delta t_c}, \end{aligned}$$

where

$$\hat{C}_h^{n,m} = C_h^{n,m}(\hat{x}^{n,m+1}), \tag{3.4.12}$$

$$\hat{x}^{n,m+1} = x - \frac{EU_h^{n,m+1}}{\Phi}\Delta t_c. \tag{3.4.13}$$

This can be done so long as $\hat{x}^{n,m+1}$ always lies in Ω. Obviously this is not the case. It is possible to define $\hat{x}^{n,m+1}$ when (3.4.13) would give a point outside of Ω, for example, by an appropriate reflection [15]. However, in petroleum reservoir simulation, the effect of the boundary is of little importance; consequently, it is reasonable to instead consider a periodic version of our problem. This has been done in most of the papers on the application of the modified method of characteristics to miscible displacement

[16], [18], [25]. We can then avoid many technical details since $\hat{x}^{n,m+1}$ is always defined in a completely natural way by (3.4.13).

We shall assume that all of our functions are Ω-periodic in x, in particular, those of $\mathcal{V} \times \mathcal{W}$ and $\mathcal{M}$. We must then drop the Neumann conditions on the differential system (2.2.3) and (2.2.5), and on the approximation space $\mathcal{V}$ (so that then $\mathcal{V} = \tilde{\mathcal{V}}_{H_P} \cap \{\psi : \psi \text{ is } \Omega\text{-periodic}\}$). The weak form of the problem in Subsection 3.3 is unchanged, except that the test functions in (3.3.1) should not have the Neumann condition. (Also, as just mentioned, all test functions should be Ω-periodic.)

3.5. A description of the approximation procedure. Our approximation procedure is defined by the following algorithm. The order of solution will be given after a description of the equations.

i) The fracture velocity and pressure:
Find $\{U_h^n, P_h^n\} \in \mathcal{V} \times \mathcal{W}$ for $n = 0, \dots, n^*$ such that

$$\big(A(C_h^n)U_h^n, \psi\big) - (\nabla \cdot \psi, P_h^n) = \big(\gamma(C_h^n), \psi\big), \qquad \psi \in \mathcal{V}, \tag{3.5.1}$$

$$(\nabla \cdot U_h^n, \varphi) = (f^n, \varphi), \qquad \varphi \in \mathcal{W}. \tag{3.5.2}$$

ii) The matrix velocity and pressure:
Find $\{u_h^n, p_h^n\} \in \mathcal{V}_i \times \mathcal{W}_i$ for each i and for $n = 0, \dots, n^*$ such that

$$\begin{aligned}\big(a(c_h^n)u_h^n, \psi\big)_i - (\nabla \cdot \psi, p_h^n)_i = \big(\gamma(c_h^n), \psi\big)_i + \big([A(C_h^m)U_h^n - \gamma(C_h^n)], \psi\big)_i \\ - (\nabla \cdot \psi, P_h^n)_i, \qquad \psi \in \mathcal{V}_i,\end{aligned} \tag{3.5.3}$$

$$(\nabla \cdot u_h^n, \varphi)_i = 0, \qquad \varphi \in \mathcal{W}_i. \tag{3.5.4}$$

We will decouple the calculations for the concentrations from those for the velocities by employing the extrapolation operator E.

iii) The fracture concentration:
Find $C_h^{n,m+1} \in \mathcal{M}$ for $n = 0, \dots, n^* - 1$ and $m = 0, \dots, m^{*,n} - 1$ such that

$$\begin{aligned}\left(\Phi\frac{C_h^{n,m+1} - \hat{C}_h^{n,m}}{\Delta t_c}, \omega\right) + \big(D(EU_h^{n,m+1})\nabla C_h^{n,m+1}, \nabla\omega\big) + (C_h^{n,m+1}f_+^{n,m+1}, \omega) \\ = (C_{inj}^{n,m+1}f_+^{n,m+1}, \omega) + \sum_i (g_{ih}^{n,m+1}, \omega), \qquad \omega \in \mathcal{M},\end{aligned} \tag{3.5.5}$$

where $\hat{C}_h^{n,m}$ is defined above by (3.4.12)–(3.4.13) and $g_{ih}^{n,m+1}$ is defined below by (3.5.13) as an affine function of $C_h^{n,m+1}$.

The computation for the matrix concentration will be split into several pieces. This will enable us to decouple it from that for the fracture concentration.

The matrix concentration problem is coupled to the fracture concentration problem through the boundary condition (2.3.7). We can approximate this condition over

$J_{n,m+1}$ as follows:

$$\begin{aligned} c^{n,m,l+1}(x) &= \Lambda_i(C^{n,m,l+1})(x) \\ &\approx \Lambda_i(IC^{n,m,l+1})(x) \\ &= \Lambda_i(C^{n,m})(x) + \sum_{j=0}^{3}(\partial C^{n,m+1}, \lambda_{i,j}\chi_i)\frac{l+1}{l^*}\Delta t_c \lambda_{i,j}(x), \qquad \text{for } x \in \partial\Omega_i. \end{aligned}$$

By solving the matrix concentration problem over $J_{n,m+1}$ with only the first piece $\Lambda_i(C^{n,m})$ of the boundary condition, we could determine the flow of fluid that would occur in the block due to the concentration distribution at time $t_{n,m}$ if no change took place on the boundary. On the other hand, we could also determine the flow of fluid over $J_{n,m+1}$ due to a unit change in the boundary condition to the value $\frac{l+1}{l^*}\Delta t_c \lambda_{i,j}(x)$ if the block originally had zero concentration everywhere. Multiplying this by the actual change $(\partial C^{n,m+1}, \lambda_{i,j}\chi_i)$ and summing on j, we would have (approximately) the flow of fluid due to the actual change in the boundary condition.

We will prescribe calculations in **iv)** and **v)** below to approximate the problems described above. In none of these problems will the fracture concentration at the advanced time level $t_{n,m+1}$ appear. The approximate matrix concentration will be defined in **vi)** in terms of $C_h^{n,m+1}$ implicitly before this quantity is known and explicitly thereafter.

We can avoid changing the coefficients of the linear systems that define the matrix concentrations at the advanced time levels during each pressure time interval by a judicious use of the extrapolation operator E [10]. As mentioned in the introduction, these linear systems should not be large, so it is appropriate to use direct solution techniques on them. In that case, a single factorization for each block for each pressure time step is needed. If, for any reason, iterative solution techniques are used instead, then the preconditioners need not be changed over the entire pressure time step.

iv) The matrix concentration assuming no change in the fracture concentration during the time interval:
Find $c_{-1h}^{n,m,l+1} \in \mathcal{M}_i^L$ for each i and for $n = 0, \ldots, n^* - 1$, $m = 0, \ldots, m^{*,n} - 1$, $l = 0, \ldots, l^* - 1$, and $j = -1$ such that

$$\left(\phi\frac{c_{jh}^{n,m,l+1} - \bar{c}_{jh}^{n,m,l}}{\delta t_c}, \omega\right)_i + (q_{jh}^{n,m,l+1}, \nabla\omega)_i = 0, \qquad \omega \in \mathcal{M}_i^0, \tag{3.5.6}$$

$$\begin{aligned} q_{jh}^{n,m,l+1} &= d(Eu_h^{n+\frac{1}{2}})\nabla c_{jh}^{n,m,l+1} - c_{jh}^{n,m,l+1} Eu_h^{n+\frac{1}{2}} \\ &\quad + \left[d(Eu_h^{n,m,l+1}) - d(Eu_h^{n+\frac{1}{2}})\right]\nabla \bar{c}_{jh}^{n,m,l} \\ &\quad - \bar{c}_{jh}^{n,m,l}\left[Eu_h^{n,m,l+1} - Eu_h^{n+\frac{1}{2}}\right], \end{aligned} \tag{3.5.7}$$

$$\bar{c}_{-1h}^{n,m,l} = \begin{cases} c_h^{n,m} & \text{if } l = 0 \\ c_{-1h}^{n,m,l} & \text{if } l \geq 1, \end{cases} \tag{3.5.8}$$

$$c_{-1h}^{n,m,l+1}(x) = \Lambda_i(C_h^{n,m})(x), \qquad \text{for } x \in \partial\Omega_i. \tag{3.5.9}$$

v) The matrix concentration correctors for unit changes in the boundary con-

dition during the time interval:
Find $c_{jh}^{n,m,l+1} \in \mathcal{M}_i^L$ for each i and for $n = 0, \ldots, n^* - 1$, $m = 0, \ldots, m^{*,n} - 1$, $l = 0, \ldots, l^* - 1$, and $j = 0, \ldots, 3$ such that (3.5.6)–(3.5.7) above hold with

$$\bar{c}_{jh}^{n,m,l} = \begin{cases} 0 & \text{if } l = 0 \\ c_{jh}^{n,m,l} & \text{if } l \geq 1, \end{cases} \tag{3.5.10}$$

$$c_{jh}^{n,m,l+1}(x) = \frac{l+1}{l^*} \lambda_{i,j}(x) \Delta t_c, \qquad \text{for } x \in \partial\Omega_i. \tag{3.5.11}$$

vi) The matrix concentration itself:
Define $c_h^{n,m+1} \in \mathcal{M}_i^L$ for each i and for $n = 0, \ldots, n^* - 1$ and $m = 0, \ldots, m^{*,n} - 1$ by

$$c_h^{n,m+1} = c_{-1h}^{n,m,l^*} + \sum_{j=0}^{3} (\partial C_h^{n,m+1}, \lambda_{i,j}\chi_i)\, c_{jh}^{n,m,l^*}. \tag{3.5.12}$$

vii) The matrix/fracture coupling:
Define for each i and for $n = 0, \ldots, n^* - 1$ and $m = 0, \ldots, m^{*,n} - 1$,

(3.5.13)

$$\begin{aligned} g_{ih}^{n,m+1} = -\frac{1}{l^*} \sum_{k=0}^{3} \sum_{l=0}^{l^*-1} \Bigg\{ & \left(\phi \frac{c_{-1h}^{n,m,l+1} - \bar{c}_{-1h}^{n,m,l}}{\delta t_c}, \lambda_{i,k} \right)_i + (q_{-1h}^{n,m,l+1}, \nabla\lambda_{i,k})_i \\ & + \sum_{j=0}^{3} (\partial C_h^{n,m+1}, \lambda_{i,j}\chi_i) \left[\left(\phi \frac{c_{jh}^{n,m,l+1} - \bar{c}_{jh}^{n,m,l}}{\delta t_c}, \lambda_{i,k} \right)_i \right. \\ & \left. + (q_{jh}^{n,m,l+1}, \nabla\lambda_{i,k})_i \right] \Bigg\} \lambda_{i,k}\chi_i. \end{aligned}$$

viii) Initialization:
Define $C_h^0 \in \mathcal{M}$ and $c_h^0 \in \mathcal{M}_i^L$ for each i in any manner such that

$$\|C^0 - C_h^0\|_0 + \|C^0 - C_h^0\|_1 H_c \leq Q \|C^0\|_S H_c^S, \qquad 1 \leq S \leq S^*, \tag{3.5.14}$$

$$\|c^0 - c_h^0\|_{0,m} + \|c^0 - c_h^0\|_{1,m} h_c \leq Q \|c^0\|_{s,m} h_c^s, \qquad 1 \leq s \leq s^*. \tag{3.5.15}$$

There are many ways to do this. One such procedure will be defined in Subsection 4.3 below.

This completes the description of the algorithm's equations, though it may be easier to conceptualize the algorithm if we include a few more. These additional equations will, in fact, be used in the convergence analysis to follow. First let us extend the definition of $c_h^{n,m+1}$ given by (3.5.12) to matrix concentration time levels. On Ω_i, let

$$c_h^{n,m,l+1} = c_{-1h}^{n,m,l+1} + \sum_{j=0}^{3} (\partial C_h^{n,m+1}, \lambda_{i,j}\chi_i)\, c_{jh}^{n,m,l+1}, \tag{3.5.16}$$

for $n = 0, \ldots, n^* - 1$, $m = 0, \ldots, m^{*,n} - 1$, and $l = 0, \ldots, l^* - 1$. Now the equations

of **iv)** and **v)** can be combined to show that $c_h^{n,m,l+1}$ satisfies the equations

$$(\phi\delta c_h^{n,m,l+1},\omega)_i + (q_h^{n,m,l+1},\nabla\omega)_i = 0, \qquad \omega\in\mathcal{M}_i^0, \tag{3.5.17}$$

$$\begin{aligned} q_h^{n,m,l+1} &= d(Eu_h^{n+\frac12})\nabla c_h^{n,m,l+1} - c_h^{n,m,l+1}Eu_h^{n+\frac12} \\ &\quad + \big[d(Eu_h^{n,m,l+1}) - d(Eu_h^{n+\frac12})\big]\nabla c_h^{n,m,l} \\ &\quad - c_h^{n,m,l}\big[Eu_h^{n,m,l+1} - Eu_h^{n+\frac12}\big], \end{aligned} \tag{3.5.18}$$

$$c_h^{n,m,l+1}(x) = \Lambda_i(IC_h^{n,m,l+1})(x), \qquad \text{for } x\in\partial\Omega_i. \tag{3.5.19}$$

Finally, then, (3.5.13) can be rewritten as

$$g_{ih}^{n,m+1} = -\sum_{k=0}^{3}\left\{(\phi\partial c_h^{n,m+1},\lambda_{i,k})_i + \frac{1}{l^*}\sum_{l=0}^{l^*-1}(q_h^{n,m,l+1},\nabla\lambda_{i,k})_i\right\}\lambda_{i,k}\chi_i, \tag{3.5.20}$$

for $n=0,\dots,n^*-1$ and $m=0,\dots,m^{*,n}-1$.

The order of solution of the algorithm will now be described. After the initialization **viii)** for C_h^0 and c_h^0, we can solve **i)** for $\{U_h^0,P_h^0\}$ and then **ii)** for $\{u_h^0,p_h^0\}$. Now we can successively step through the pressure time levels. For the nth, we must first determine the concentrations at the fracture concentration time levels. This is done for the mth by solving **iv)** and **v)**, $l=0,\dots,l^*-1$, for the five pieces of the matrix concentration $c_{jh}^{n,m,l+1}$, $j=-1,\dots,3$. With these, **iii)** and **vii)** must be combined and solved to obtain $C_h^{n,m+1}$. Once $C_h^{n,m+1}$ is known, **vi)** gives us $c_h^{n,m+1}$. Finally, when the fracture concentration time level has progressed to the $(n+1)$-st pressure time level, we can again use **i)** and **ii)** to solve for the velocities and pressures U_h^{n+1}, P_h^{n+1}, u_h^{n+1}, and p_h^{n+1}, completing the pressure time step.

3.6. Implementation of the procedure. We end this section with two brief remarks on implementing the approximation procedure.

In practice, it is not necessary to know the state of the matrix fluid in each block. The pattern of flow (and, more importantly, the total amount of oil) in an individual block can be inferred from the situation in nearby blocks. Consequently, it is sufficient to calculate approximate solutions to the matrix equations on a representative set of blocks. Such a set of blocks can be determined from the fracture equations. In any field-scale simulation, the spatial partition $\mathcal{T}_{H_c}(\Omega)$ will necessarily be coarser than the partition of Ω arising from the physical fractures. That is, every finite element of the fracture system will sit over many matrix blocks. A reasonable choice for the representative set of blocks consists only of those that contain the quadrature points of the fracture concentration calculation. This, then, significantly reduces the number of equations that need to be solved.

As in the case of simulating the flow of a single component in a single phase, the approximation procedure is well suited to solution on a parallel computing network [1], [11], since the block equations are independent of each other and there is relatively little transfer of data between the fracture and matrix calculations.

§4. An Analysis of the Convergence of the Procedure

In this section we give an asymptotic analysis of the convergence of the solutions of the approximation procedure to smooth solutions of the differential model. The quantities that have been assumed to be smooth in Subsection 3.2 above will be taken as such without further comment. We begin by stating two theorems that describe the convergence results. The rest of the section is devoted to their proofs. In Subsection 4.2, we derive equations for the pressure and velocity errors and bound them in terms of the concentration errors. These concentration errors are systematically analyzed in the last five subsections. It is often valuable to analyze parabolic equations by first considering the error associated to an elliptic projection of the true solution [30]. This we do in Subsection 4.3. Once this is done, in the following subsection we can derive an equation for the projection of the fracture concentration error and bound all of the terms that are not coupled to the matrix. Similarly, in the subsection after that, an equation for the projection of the matrix concentration error is derived and all terms not coupled to the fracture system are bounded. Then, in Subsection 4.6, we obtain a bound for these coupling terms. Finally, an induction argument is applied in Subsection 4.7 to control the effects of some of the nonlinearities, completing the analysis of the concentration errors.

4.1. Statement of the convergence results. For the appropriate R, r, S, and s, let the main error be represented by

(4.1.1)
$$\begin{aligned}\mathcal{E} = &\left[\left\|\frac{\partial^2 C}{\partial \tau_E^2}\right\|_{L^2(L^2)} + \|u_t\|_{L^2(L^2_m)} + \|c\|_{H^1(H^1_m)} + \|c_{tt}\|_{L^2(L^2_m)}\right]\Delta t_c \\ &+ [\,\|U_{tt}\|_{L^2(L^2)} + \|u_{tt}\|_{L^2(L^2_m)}](\Delta t_p)^2 \\ &+ [\,\|U_t\|_{L^2(J_1;L^2)} + \|u_t\|_{L^2(J_1;L^2_m)}]\Delta t_p^1 \\ &+ \|U\|_{L^\infty(H^R)} H_p^R + \|u\|_{L^\infty(H^r_m)} h_p^r \\ &+ [\,\|C\|_{L^\infty(H^S)} + \|C\|_{H^1(H^S)}] H_c^S \\ &+ [\,\|c\|_{L^\infty(H^s_m)} + \|c\|_{H^1(H^s_m)}] h_c^s,\end{aligned}$$

where τ_E is defined below in (4.4.8).

THEOREM 4.1.1. *If $\Delta t_c h_c^{-2} = o(1)$ as $\Delta t_c, h_c \longrightarrow 0$, and if asymptotically*

$$\big(H_p^R + h_p^r + (\Delta t_p)^2 + \Delta t_p^1 + H_c^S + h_c^s + \Delta t_c\big)\big(H_c^{-\frac{3}{2}} + h_c^{-\frac{3}{2}}\big) = o(1)$$

as the discretization parameters tend to zero for some R, r, S, and s such that $1 \le R \le R^*$, $1 \le r \le r^*$, $2 \le S \le S^*$, and $2 \le s \le s^*$, then for sufficiently small discretization parameters, the following estimate holds:

$$\max_{n,m}\big\{\|C^{n,m} - C_h^{n,m}\|_0 + \|c^{n,m} - c_h^{n,m}\|_{0,m} + \|U^n - U_h^n\|_0 + \|u^n - u_h^n\|_{0,m}\big\} \le Q\mathcal{E}.$$

Moreover,

$$\Big(\sum_{n,m}^{n^*-1,m^*-1} \|\nabla(C-C_h)^{n,m+1}\|_0^2 \Delta t_c\Big)^{\frac{1}{2}} + \Big(\sum_{n,m}^{n^*-1,m^*-1}\sum_{l=0}^{l^*-1} \|\nabla(c-c_h)^{n,m,l+1}\|_{0,m}^2 \delta t_c\Big)^{\frac{1}{2}} \le Q\big\{\mathcal{E} + \|C\|_{L^\infty(H^{S'+1})} H_c^{S'} + \|c\|_{L^\infty(H_m^{s'+1})} h_c^{s'}\big\},$$

where $S' = min(S, S^*-1)$ *and* $s' = min(s, s^*-1)$, *and*

$$\max_n\{\|P^n - P_h^n\|_0 + \|p^n - p_h^n\|_{0,m}\} \le Q\big\{\mathcal{E} + \|P\|_{L^\infty(H^{R'})} H_p^{R'} + \|p\|_{L^\infty(H_m^{r'})} h_p^{r'}\big\},$$

where $R' = min(R, R^{**})$ *and* $r' = min(r, r^{**})$.

THEOREM 4.1.2. *For any* R *such that* $0 \le R \le R^{**}$,

$$\max_n \|\nabla\cdot(U^n - U_h^n)\|_0 \le Q\,\|\nabla\cdot U\|_{L^\infty(H^R)} H_p^R,$$

and, for all n,

$$\nabla\cdot(u^n - u_h^n) = 0.$$

Since $S^* \ge 2$ and $s^* \ge 2$, the assumptions of the first theorem are quite reasonable. They merely say that H_c and h_c cannot tend to zero too fast compared to the other parameters. These two convergence theorems are a combination of (4.7.7) and Lemmas 4.2.2 and 4.3.1 below.

4.2. Analysis of the pressure error equations. For $n = 0, \dots, n^*$, let

$$\Upsilon^n = U^n - U_h^n, \qquad \Theta^n = P^n - P_h^n,$$
$$\upsilon^n = u^n - u_h^n, \qquad \theta^n = p^n - p_h^n.$$

These errors satisfy a set of equations given by subtracting (3.5.1)–(3.5.4) from an evaluation of (3.3.1)–(3.3.4) at the time t_n:

$$\begin{aligned}(A(C_h^n)\Upsilon^n,\psi) - (\nabla\cdot\psi,\Theta^n) &= \big([\gamma(C^n) - \gamma(C_h^n)],\psi\big) \\ &\quad - \big([A(C^n) - A(C_h^n)]U^n,\psi\big), \qquad \psi\in\mathcal{V},\end{aligned} \tag{4.2.1}$$

$$(\nabla\cdot\Upsilon^n,\varphi) = 0, \qquad \varphi\in\mathcal{W}, \tag{4.2.2}$$

$$\begin{aligned}\big(a(c_h^n)\upsilon^n,\psi\big)_i - (\nabla\cdot\psi,\theta^n)_i &= \big([\gamma(c^n) - \gamma(c_h^n)],\psi\big)_i - (\nabla\cdot\psi,\Theta^n)_i \\ &\quad + \big(A(C_h^n)\Upsilon^n,\psi\big)_i - \big([\gamma(C^n) - \gamma(C_h^n)],\psi\big)_i \\ &\quad - \big([a(c^n) - a(c_h^n)]u^n,\psi\big)_i \\ &\quad + \big([A(C^n) - A(C_h^n)]U^n,\psi\big)_i, \qquad \psi\in\mathcal{V}_i,\end{aligned} \tag{4.2.3}$$

$$(\nabla\cdot\upsilon^n,\varphi)_i = 0, \qquad \varphi\in\mathcal{W}_i. \tag{4.2.4}$$

Equations (4.2.1)–(4.2.2) have been analyzed in [9] by making use of a certain projection of $\{U^n, P^n\}$ into $\mathcal{V}\times\mathcal{W}$. The result can be refined as described in [4] by making use of the duality ideas of Douglas and Roberts [12]. The refined estimates show that L^2-norms of the errors Υ^n and Θ^n are bounded by the sum of H_p to the optimal power and $\|C^n - C_h^n\|_0$, where the regularity needed of the solution is optimal and

where quasi-regularity of the partition $\mathcal{T}_{H_p}(\Omega)$ is required to bound the L^∞-norm of the projection of U^n.

We will use a more direct approach to analyze (4.2.1)–(4.2.4). Standard projections of $\{U^n, P^n\}$ into $\mathcal{V} \times \mathcal{W}$ and of $\{u^n, p^n\}$ into $\mathcal{V}_i \times \mathcal{W}_i$ will be used rather than the special projections of [9]. The advantage of this approach is that it is easy to understand and it does not require that the partition be quasi-regular. Let us start by considering the situation on $\mathcal{V} \times \mathcal{W}$.

With H_N^1 denoting the space $H_N(div; \Omega) \cap (H^1)^3$, each of our mixed finite element spaces [6], [7], [22], [24] has the property that there exist projection operators $\Pi : H_N^1 + \mathcal{V} \longrightarrow \mathcal{V}$ and $\mathcal{P} = L^2$-projection : $L_C^2 \longrightarrow \mathcal{W}$ such that for $(v, w) \in H_N^1 \times L_C^2$,

$$\|v - \Pi v\|_0 \le Q \|v\|_R H_p^R, \qquad 1 \le R \le R^*, \tag{4.2.5}$$

$$\|\nabla \cdot (v - \Pi v)\|_0 \le Q \|\nabla \cdot v\|_R H_p^R, \qquad 0 \le R \le R^{**}, \tag{4.2.6}$$

$$\|w - \mathcal{P} w\|_0 \le Q \|w\|_R H_p^R, \qquad 0 \le R \le R^{**}, \tag{4.2.7}$$

and, on $\mathcal{V}$,

$$div \Pi = \mathcal{P} div. \tag{4.2.8}$$

Also, $div \mathcal{V} = \mathcal{W}$; in fact:

LEMMA 4.2.1. *Given $w \in \mathcal{W}$, there exists $v \in \mathcal{V}$ such that*

$$\nabla \cdot v = w \qquad \text{and} \qquad \|v\|_{H(div;\Omega)} \le Q \|w\|_0,$$

where the $H(div; \Omega)$-norm is given by

$$\|v\|_{H(div;\Omega)} = \{ \|v\|_0^2 + \|\nabla \cdot v\|_0^2 \}^{1/2}.$$

This lemma was proven by Raviart and Thomas [24, Theorem 4] for the spaces of their paper. The lemma is true for any of the spaces $\mathcal{V}$ and $\mathcal{W}$ satisfying the properties mentioned above. For the sake of completeness, we present a simple proof below. A more general version of this kind of proof appears in [12].

Proof. Modulo the constant functions, one can solve the elliptic system

$$\begin{aligned} \Delta \varphi &= w \qquad \text{in } \Omega, \\ \nabla \varphi \cdot \nu &= 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

It is well known that

$$\|\varphi\|_2 \le Q \|w\|_0.$$

Let

$$v = \Pi \nabla \varphi \in \mathcal{V}.$$

Then, by (4.2.8),

$$\nabla \cdot v = \nabla \cdot \Pi \nabla \varphi = \mathcal{P} \Delta \varphi = \mathcal{P} w = w \in \mathcal{W}$$

and, by (4.2.5),

$$\|v\|_0 = \|\Pi \nabla \varphi\|_0 \le \|\Pi \nabla \varphi - \nabla \varphi\|_0 + \|\nabla \varphi\|_0 \le Q \|\varphi\|_2 H_p + \|\varphi\|_1 \le Q \|w\|_0.$$

Analogous properties hold for the projections Π_i and $\mathcal{P}_i$ associated to $\mathcal{V}_i \times \mathcal{W}_i$.

With these preliminaries out of the way, it is easy to analyze (4.2.1)–(4.2.2). First, take $\psi = \Pi\Upsilon^n$ in (4.2.1) and $\varphi = \mathcal{P}\Theta^n$ in (4.2.2). Add the two equations together to see that

$$\begin{aligned}\big(A(C_h^n)\Upsilon^n, \Pi\Upsilon^n\big) - (\nabla\cdot\Pi\Upsilon^n, \Theta^n) + (\nabla\cdot\Upsilon^n, \mathcal{P}\Theta^n)&\\ = \big([\gamma(C^n) - \gamma(C_h^n)], \Pi\Upsilon^n\big) - \big([A(C^n) - A(C_h^n)]U^n, \Pi\Upsilon^n\big)&\\ \le Q\|C^n - C_h^n\|_0^2 + \epsilon\|\Pi\Upsilon^n\|_0^2.&\end{aligned}$$

The last two terms on the far left-hand side above cancel by (4.2.8) (since $\mathcal{P}$ is L^2-projection). Consequently, since $\Upsilon^n - \Pi\Upsilon^n = U^n - \Pi U^n$,

$$\begin{aligned}\|\Upsilon^n\|_0^2 &\le Q\big\{\big(A(C_h^n)\Upsilon^n, \Pi\Upsilon^n\big) + \big(A(C_h^n)\Upsilon^n, U^n - \Pi U^n\big)\big\}\\ &\le Q\big\{\|U^n - \Pi U^n\|_0^2 + \|C^n - C_h^n\|_0^2\big\} + \epsilon\|\Upsilon^n\|_0^2.\end{aligned} \tag{4.2.9}$$

Next, (4.2.2) shows that $\nabla\cdot U_h^n = \mathcal{P}\nabla\cdot U^n$ $(= \mathcal{P}f^n)$, so

$$\|\nabla\cdot\Upsilon^n\|_0 = \|\nabla\cdot U^n - \mathcal{P}\nabla\cdot U^n\|_0. \tag{4.2.10}$$

Finally, in (4.2.1) let ψ be the function associated to $\mathcal{P}\Theta^n$ by Lemma 4.2.1. Then

$$\begin{aligned}(\mathcal{P}\Theta^n, \Theta^n) = (\nabla\cdot\psi, \Theta^n) &= \big(A(C_h^n)\Upsilon^n, \psi\big) - \big([\gamma(C^n) - \gamma(C_h^n)], \psi\big)\\ &\quad + \big([A(C^n) - A(C_h^n)]U^n, \psi\big)\\ &\le Q\big\{\|\Upsilon^n\|_0^2 + \|C^n - C_h^n\|_0^2\big\} + \epsilon\|\psi\|_0^2\\ &\le Q\big\{\|\Upsilon^n\|_0^2 + \|C^n - C_h^n\|_0^2\big\} + \epsilon\|\mathcal{P}\Theta^n\|_0^2.\end{aligned}$$

Since $\Theta^n - \mathcal{P}\Theta^n = P^n - \mathcal{P}P^n$ and $\mathcal{P}$ is bounded on L^2,

$$\begin{aligned}\|\Theta^n\|_0^2 &= (P^n - \mathcal{P}P^n, \Theta^n) + (\mathcal{P}\Theta^n, \Theta^n)\\ &\le Q\big\{\|P^n - \mathcal{P}P^n\|_0^2 + \|\Upsilon^n\|_0^2 + \|C^n - C_h^n\|_0^2\big\} + \epsilon\|\Theta^n\|_0^2.\end{aligned} \tag{4.2.11}$$

Analogously, we can analyze (4.2.3)–(4.2.4) to derive the following:

$$\begin{aligned}\|v^n\|_{0,i}^2 \le Q\big\{&\|u^n - \Pi_i u^n\|_{0,i}^2 + \|c^n - c_h^n\|_{0,i}^2\\ &+ \|\Upsilon^n\|_{0,i}^2 + \|C^n - C_h^n\|_{0,i}^2\big\},\end{aligned} \tag{4.2.12}$$

$$\nabla\cdot v^n = 0, \tag{4.2.13}$$

$$\begin{aligned}\|\theta^n\|_{0,i}^2 \le Q\big\{&\|p^n - \mathcal{P}_i p^n\|_{0,i}^2 + \|v^n\|_{0,i}^2 + \|c^n - c_h^n\|_{0,i}^2\\ &+ \|\Theta^n\|_{0,i}^2 + \|\Upsilon^n\|_{0,i}^2 + \|C^n - C_h^n\|_{0,i}^2\big\}.\end{aligned} \tag{4.2.14}$$

Combining (4.2.9)–(4.2.14) with (4.2.5)–(4.2.7) and the analogous estimates on the $\mathcal{V}_i \times \mathcal{W}_i$ gives the following lemma.

LEMMA 4.2.2. *For $n = 0, \dots, n^*$,*

$$\begin{aligned}
\|\Upsilon^n\|_0 &\le Q\big\{\|U^n\|_R H_p^R + \|C^n - C_h^n\|_0\big\}, && 1 \le R \le R^*,\\
\|\nabla\cdot\Upsilon^n\|_0 &\le Q\,\|\nabla\cdot U^n\|_R H_p^R, && 0 \le R \le R^{**},\\
\|\Theta^n\|_0 &\le Q\big\{\|P^n\|_R H_p^R + \|\Upsilon^n\|_0 + \|C^n - C_h^n\|_0\big\}, && 0 \le R \le R^{**},
\end{aligned}$$

$$\begin{aligned}
\|\upsilon^n\|_{0,m} &\le Q\big\{\|u^n\|_{r,m} h_p^r + \|c^n - c_h^n\|_{0,m}\\
&\qquad + \|U^n\|_R H_p^R + \|C^n - C_h^n\|_0\big\}, && 1 \le r \le r^*,\ 1 \le R \le R^*,\\
\nabla\cdot\upsilon^n &= 0, && \text{on } \Omega_m,\\
\|\theta^n\|_{0,m} &\le Q\big\{\|p^n\|_{r,m} h_p^r + \|\upsilon^n\|_{0,m} + \|c^n - c_h^n\|_{0,m}\\
&\qquad + \|P^n\|_R H_p^R + \|\Upsilon^n\|_0 + \|C^n - C_h^n\|_0\big\}, && 0 \le r \le r^{**},\ 0 \le R \le R^{**}.
\end{aligned}$$

The nonoptimal regularity required of the pressures appears to be avoided in [4] at the expense of requiring quasi-regularity of the partition. However, the concentration estimates will depend on the velocity errors, so this additional regularity is needed in any case.

4.3. Some projections of the concentrations. Following [9], let $\tilde{C} : J \longrightarrow \mathcal{M}$ be defined by

$$\begin{aligned}
&\big(D(U)\nabla\tilde{C}, \nabla\omega\big) + \big((\sigma + f_+)\tilde{C}, \omega\big)\\
&\qquad = \big(D(U)\nabla C, \nabla\omega\big) + \big((\sigma + f_+)C, \omega\big)\\
&\qquad = -(\Phi C_t + U\cdot\nabla C, \omega) + (C_{inj} f_+, \omega)\\
&\qquad\qquad + \sum_i \big(g_i(c,u), \omega\big) + (\sigma C, \omega), \qquad \omega \in \mathcal{M},
\end{aligned} \tag{4.3.1}$$

where σ is large enough to ensure coercivity of the elliptic form ($\sigma \ge \sigma_* > 0$ suffices), and where the last equality above is from (3.3.5).

Similarly, on each Ω_i, let $\tilde{c} : J \longrightarrow \mathcal{M}_i^L$ be defined by

$$\big(q(\tilde{c}, u), \nabla\omega\big)_i = \big(q(c,u), \nabla\omega\big)_i = -(\phi c_t, \omega)_i, \qquad \omega \in \mathcal{M}_i^0, \tag{4.3.2}$$

$$\tilde{c}(x,t) = c(x,t) = \Lambda_i\big(C(\,\cdot\,,t)\big)(x), \qquad \text{for } (x,t) \in \partial\Omega_i \times J, \tag{4.3.3}$$

where we have used (3.3.7)–(3.3.8). Recall that q is defined by (2.3.4), so the form in (4.3.2) is elliptic. It is also coercive on $H_0^1(\Omega_i)$ since $\nabla\cdot u = 0$.

The usual analyses of finite element methods for elliptic problems show that

LEMMA 4.3.1.

$$\begin{aligned}
\|C - \tilde{C}\|_0 \ +\ \|C - \tilde{C}\|_1 H_c &\le Q\|C\|_S H_c^S, && 1 \le S \le S^*,\\
\left\|\frac{\partial(C - \tilde{C})}{\partial t}\right\|_0 + \left\|\frac{\partial(C - \tilde{C})}{\partial t}\right\|_1 H_c &\le Q\big\{\|C\|_S + \|C_t\|_S\big\} H_c^S, && 1 \le S \le S^*,
\end{aligned}$$

$$\begin{aligned}
\|c - \tilde{c}\|_{0,m} \ +\ \|c - \tilde{c}\|_{1,m} h_c &\le Q\|c\|_{s,m} h_c^s, && 1 \le\ s \le s^*,\\
\left\|\frac{\partial(c - \tilde{c})}{\partial t}\right\|_{0,m} + \left\|\frac{\partial(c - \tilde{c})}{\partial t}\right\|_{1,m} h_c &\le Q\big\{\|c\|_{s,m} + \|c_t\|_{s,m}\big\} h_c^s, && 1 \le\ s \le s^*.
\end{aligned}$$

The function $\Lambda_i(C)$ does not appear in the norms on the right-hand sides of the last two estimates above since $\|c - \Lambda_i(C)\|_{1,m} \le Q\|c\|_{1,m}$ [2].

It is also possible to show that, if the solution is sufficiently smooth, the $W^{1,\infty}$-norms of $\tilde{C}$ and $\tilde{c}$ are bounded (since the concentration partitions are quasi-regular):

$$\|\tilde{C}\|_{1,\infty} + \|\tilde{c}\|_{1,\infty,m} \le Q. \tag{4.3.4}$$

Let

$$\Xi^{n,m} = \tilde{C}^{n,m} - C_h^{n,m} \qquad \text{and} \qquad \xi^{n,m,l} = I\tilde{c}^{n,m,l} - c_h^{n,m,l};$$

it remains to estimate the sizes of these errors.

Note that Lemmas 4.2.2 and 4.3.1 together imply that

$$\sum_{n,m}^{N,M} \|\Upsilon^n\|_0^2 \Delta t_c \le Q\Big\{\mathcal{E}^2 + \sum_{n,m}^{N,M} \|\Xi^n\|_0^2 \Delta t_c\Big\}, \tag{4.3.5}$$

$$\sum_{n,m}^{N,M} \|v^n\|_{0,m}^2 \Delta t_c \le Q\Big\{\mathcal{E}^2 + \sum_{n,m}^{N,M} \big(\|\xi^n\|_{0,m}^2 + \|\Xi^n\|_0^2\big)\Delta t_c\Big\}, \tag{4.3.6}$$

for any N and M.

In the rest of the analysis, we shall tacitly assume that

$$C_h^0 = \tilde{C}^0 \qquad \text{and} \qquad c_h^0 = \tilde{c}^0$$

so that

$$\Xi^0 = 0 \qquad \text{and} \qquad \xi^0 = 0.$$

By (3.5.14)–(3.5.15), any other reasonable initialization will have the same final convergence estimate.

4.4. An analysis of the fracture concentration error equation. We will now analyze those terms that arise in the fracture concentration error equation that are not coupled to the matrix quantities. Such analyses have appeared elsewhere [13], [16], [18], [25]. For our purposes, the paper of Ewing, Russell, and Wheeler [18] is the most convenient. Their work requires an asymptotic restriction that we shall remove in a manner that is based on the work of Durán [16].

The above authors make use of evaluations of functions at not only $\hat{x}$ (defined in (3.4.13) above), but also at $\check{x}$, where

$$\check{x}^{n,m+1} = x - \frac{EU^{n,m+1}}{\Phi}\Delta t_c. \tag{4.4.1}$$

Let

$$\hat{\omega}^{n,m} = \omega^{n,m}(\hat{x}^{n,m+1}) = \omega(\hat{x}^{n,m+1}, t_{n,m}), \tag{4.4.2}$$

$$\check{\omega}^{n,m} = \omega^{n,m}(\check{x}^{n,m+1}) = \omega(\check{x}^{n,m+1}, t_{n,m}). \tag{4.4.3}$$

An equation for the error $\Xi^{n,m+1}$ is obtained by combining (3.5.5) with an evaluation

of (4.3.1) at time $t_{n,m+1}$. We have after some manipulation that

(4.4.4)
$$
\begin{aligned}
&(\Phi\partial\Xi^{n,m+1},\omega)+\big(D(EU_h^{n,m+1})\nabla\Xi^{n,m+1},\nabla\omega\big)+(\Xi^{n,m+1}f_+^{n,m+1},\omega)\\
&\quad=-\big([D(U^{n,m+1})-D(EU_h^{n,m+1})]\nabla\tilde{C}^{n,m+1},\nabla\omega\big)+\big(\sigma(C^{n,m+1}-\tilde{C}^{n,m+1}),\omega\big)\\
&\qquad-\big(\Phi(\partial C^{n,m+1}-\partial\tilde{C}^{n,m+1}),\omega\big)-\big([U^{n,m+1}-EU^{n,m+1}]\cdot\nabla C^{n,m+1},\omega\big)\\
&\qquad-\left([\Phi C_t^{n,m+1}+EU^{n,m+1}\cdot\nabla C^{n,m+1}]-\Phi\frac{C^{n,m+1}-\check{C}^{n,m}}{\Delta t_c},\omega\right)\\
&\qquad+\left(\Phi\frac{(\check{C}-\check{\tilde{C}})^{n,m}-(C-\tilde{C})^{n,m}}{\Delta t_c},\omega\right)+\left(\Phi\frac{\check{\Xi}^{n,m}-\Xi^{n,m}}{\Delta t_c},\omega\right)\\
&\qquad-\left(\Phi\frac{\hat{\check{C}}^{n,m}-\check{\tilde{C}}^{n,m}}{\Delta t_c},\omega\right)+\left(\Phi\frac{\hat{\Xi}^{n,m}-\check{\Xi}^{n,m}}{\Delta t_c},\omega\right)\\
&\qquad+\sum_i\big(g_i^{n,m+1}-g_{ih}^{n,m+1},\omega\big),\qquad\omega\in\mathcal{M}.
\end{aligned}
$$

We choose the test function $\omega=\Xi^{n,m+1}\in\mathcal{M}$ in the error equation. The left-hand side of (4.4.4) is bounded from below by the expression

$$
\frac{1}{2\Delta t_c}[(\Phi\Xi^{n,m+1},\Xi^{n,m+1})-(\Phi\Xi^{n,m},\Xi^{n,m})]+D_*\|\nabla\Xi^{n,m+1}\|_0^2 \tag{4.4.5}
$$

The first four terms on the right-hand side of (4.4.4) are easily dealt with (see [9] and [10]). Firstly, we have for $n\geq 1$ that

$$
\begin{aligned}
&-\big([D(U^{n,m+1})-D(EU_h^{n,m+1})]\nabla\tilde{C}^{n,m+1},\nabla\Xi^{n,m+1}\big)\\
&\qquad\qquad-\big([U^{n,m+1}-EU^{n,m+1}]\cdot\nabla C^{n,m+1},\Xi^{n,m+1}\big)\\
&\qquad\leq Q\big\{\|U_{tt}\|^2_{L^2(J_{n+1}\cap J_n;L^2)}(\Delta t_p)^3+\|E\Upsilon^{n,m+1}\|_0^2+\|\Xi^{n,m+1}\|_0^2\big\}\\
&\qquad\qquad+\epsilon\,\|\nabla\Xi^{n,m+1}\|_0^2\\
&\qquad\leq Q\big\{\|U_{tt}\|^2_{L^2(J_{n+1}\cap J_n;L^2)}(\Delta t_p)^3+\|\Upsilon^{n}\|_0^2+\|\Upsilon^{n-1}\|_0^2+\|\Xi^{n,m+1}\|_0^2\big\}\\
&\qquad\qquad+\epsilon\,\|\nabla\Xi^{n,m+1}\|_0^2,
\end{aligned}
\tag{4.4.6}
$$

where we have used (4.3.4) and the Lipschitz character of $D(U)$ (see [9, (7.4)]). If $n=0$, then we have the same bounds except that $\|U_{tt}\|^2_{L^2(J_{n+1}\cap J_n;L^2)}(\Delta t_p)^3$ must be replaced by $\|U_t\|^2_{L^2(J_1;L^2)}\Delta t_p^1$ since the extrapolation operator E is only first order accurate on J_1, and we must interpret the term at time level t_{n-1} as zero. Secondly, with Lemma 4.3.1,

(4.4.7)
$$
\begin{aligned}
&\big(\sigma(C^{n,m+1}-\tilde{C}^{n,m+1}),\Xi^{n,m+1}\big)-\big(\Phi(\partial C^{n,m+1}-\partial\tilde{C}^{n,m+1}),\Xi^{n,m+1}\big)\\
&\qquad\leq Q\big\{\|C^{n,m+1}-\tilde{C}^{n,m+1}\|_0^2+\|C_t-\tilde{C}_t\|^2_{L^2(J_{n,m+1};L^2)}(\Delta t_c)^{-1}+\|\Xi^{n,m+1}\|_0^2\big\}\\
&\qquad\leq Q\big\{\|C\|^2_{L^\infty(H^S)}H_c^{2S}+\|C\|^2_{H^1(J_{n,m+1};H^S)}(\Delta t_c)^{-1}H_c^{2S}+\|\Xi^{n,m+1}\|_0^2\big\}.
\end{aligned}
$$

The next three terms are estimated with the aid of a change of variables. In [13], [18], and [25] it is shown that the mapping $x \longmapsto \check{x}^{n,m+1}$ is, by periodicity, a differentiable homeomorphism of Ω onto itself for Δt_c small enough. The determinant of the Jacobian is $1 + O(\Delta t_c)$. Hence, integrals involving $\check{x}^{n,m+1}$ can be converted into integrals involving x alone quite readily.

With the approximate characteristic direction

$$\tau_E(x,t) = \frac{(EU^{n,m+1}, \Phi)}{\sqrt{|EU^{n,m+1}|^2 + \Phi^2}} \approx \tau(x,t) \tag{4.4.8}$$

for $t \in J_{n,m+1}$, it is known that the fifth term on the right-hand side of (4.4.4) satisfies

$$\begin{aligned} &-\left([\Phi C_t^{n,m+1} + EU^{n,m+1} \cdot \nabla C^{n,m+1}] - \Phi \frac{C^{n,m+1} - \check{C}^{n,m}}{\Delta t_c}, \Xi^{n,m+1}\right) \\ &\quad = -\left(\sqrt{|EU^{n,m+1}|^2 + \Phi^2} \frac{\partial C^{n,m+1}}{\partial \tau_E} - \Phi \frac{C^{n,m+1} - \check{C}^{n,m}}{\Delta t_c}, \Xi^{n,m+1}\right) \qquad (4.4.9) \\ &\quad \le Q\left\{ \left\| \frac{\partial^2 C}{\partial \tau_E^2} \right\|^2_{L^2(J_{n,m+1};L^2)} \Delta t_c + \|\Xi^{n,m+1}\|_0^2 \right\}, \end{aligned}$$

since $\Phi \frac{C^{n,m+1} - \check{C}^{n,m}}{\Delta t_c}$ is essentially a backward difference approximation to the derivative $\sqrt{|EU^{n,m+1}|^2 + \Phi^2} \frac{\partial C^{n,m+1}}{\partial \tau_E}$. It is (4.4.9) that shows the value of the modified method of characteristics. The analysis shows that the bound on the fracture time truncation error is proportional to $\left\| \frac{\partial^2 C}{\partial \tau_E^2} \right\|_{L^2(L^2)} \Delta t_c$; whereas, in a more standard Galerkin approximation procedure, this bound is proportional to $\|C_{tt}\|_{L^2(L^2)} \Delta t_c$. Because the fracture concentration equation is convection dominated, $\frac{\partial C}{\partial \tau} \approx 0$, so we expect that

$$\left| \frac{\partial^2 C}{\partial \tau_E^2} \right| \ll |C_{tt}|.$$

Hence, in theory it should be acceptable to take a reasonably long fracture concentration time step; this has been observed in practice for miscible displacement in an unfractured reservoir [17].

To obtain an optimal order estimate of the sixth term on the right-hand side of (4.4.4), the expression

$$\frac{(\check{C} - \check{\tilde{C}})^{n,m} - (C - \tilde{C})^{n,m}}{\Delta t_c},$$

which acts as if it were a spatial derivative, is estimated in the norm of the dual space of H^1. A change of variables argument shows that

$$\begin{aligned} -\left(\Phi \frac{(\check{C} - \check{\tilde{C}})^{n,m} - (C - \tilde{C})^{n,m}}{\Delta t_c}, \Xi^{n,m+1}\right) &\le Q\|(C - \tilde{C})^{n,m}\|_0^2 + \epsilon \|\Xi^{n,m+1}\|_1^2 \\ &\le Q\|C\|^2_{L^\infty(H^S)} H_c^{2S} + \epsilon \|\Xi^{n,m+1}\|_1^2, \end{aligned} \tag{4.4.10}$$

using Lemma 4.3.1. Similarly, the seventh term satisfies

$$\left(\Phi \frac{\check{\Xi}^{n,m} - \Xi^{n,m}}{\Delta t_c}, \Xi^{n,m+1}\right) \le Q\|\Xi^{n,m}\|_0^2 + \epsilon \|\Xi^{n,m+1}\|_1^2. \tag{4.4.11}$$

The eighth and ninth terms on the right-hand side of (4.4.4) can be bounded easily. The aforementioned papers contain the ideas needed for the simple estimate that

$$
\begin{aligned}
&\left|\left(\Phi\frac{\hat{\varphi}^{n,m}-\check{\varphi}^{n,m}}{\Delta t_c},\psi\right)\right| \\
&\quad=\left|\left(\int_0^1\nabla\varphi^{n,m}\big(\check{x}^{n,m+1}+\frac{E\Upsilon^{n,m+1}}{\Phi}\Delta t_c s\big)\cdot E\Upsilon^{n,m+1}\,ds,\psi\right)\right| \\
&\quad\le\|\nabla\varphi^{n,m}\|_{0,\infty}\,\|E\Upsilon^{n,m+1}\|_0\,\|\psi\|_0.
\end{aligned}
\tag{4.4.12}
$$

Hence, with (4.3.4) and an inverse inequality,

$$
\begin{aligned}
&-\left(\Phi\frac{\hat{\tilde{C}}^{n,m}-\check{\tilde{C}}^{n,m}}{\Delta t_c},\Xi^{n,m+1}\right)+\left(\Phi\frac{\hat{\Xi}^{n,m}-\check{\Xi}^{n,m}}{\Delta t_c},\Xi^{n,m+1}\right) \\
&\quad\le[\,\|\nabla\tilde{C}^{n,m}\|_{0,\infty}+\|\nabla\Xi^{n,m}\|_{0,\infty}\,]\|E\Upsilon^{n,m+1}\|_0\,\|\Xi^{n,m+1}\|_0 \\
&\quad\le Q[1+\|\nabla\Xi^{n,m}\|_0 H_c^{-\frac{3}{2}}][\,\|\Upsilon^n\|_0+\|\Upsilon^{n-1}\|_0]\|\Xi^{n,m+1}\|_0 \\
&\quad\le\epsilon(\|\Upsilon^n\|_0^2+\|\Upsilon^{n-1}\|_0^2)H_c^{-3}\|\nabla\Xi^{n,m}\|_0^2 \\
&\qquad+Q\big\{\|\Upsilon^n\|_0^2+\|\Upsilon^{n-1}\|_0^2+\|\Xi^{n,m+1}\|_0^2\big\}.
\end{aligned}
\tag{4.4.13}
$$

Upon summing (4.4.4) on n and m, we obtain from the above estimates and (4.3.5) the result that

$$
\begin{aligned}
&\frac{1}{2}\Phi_*\|\Xi^{N,M+1}\|_0^2+(D_*-\epsilon)\sum_{n,m}^{N,M}\|\nabla\Xi^{n,m+1}\|_0^2\Delta t_c \\
&\quad\le Q\left\{\mathcal{E}^2+\sum_{n,m}^{N,M}\|\Xi^{n,m+1}\|_0^2\Delta t_c\right\} \\
&\qquad+\epsilon\sum_{n,m}^{N,M}(\|\Upsilon^n\|_0^2+\|\Upsilon^{n-1}\|_0^2)H_c^{-3}\|\nabla\Xi^{n,m}\|_0^2\Delta t_c \\
&\qquad+\sum_{n,m}^{N,M}\sum_i\big(g_i^{n,m+1}-g_{ih}^{n,m+1},\Xi^{n,m+1}\big)\Delta t_c.
\end{aligned}
\tag{4.4.14}
$$

Only the last term on the right-hand side above needs to be estimated. It is coupled to the matrix quantities, so we leave its estimation to a later subsection.

4.5. An analysis of the matrix concentration error equations. An equation for $\xi^{n,m,l+1}$ is obtained by combining (3.5.17) with the interpolation of (4.3.2) at

times $t_{n,m}$ and $t_{n,m+1}$ to the time $t_{n,m,l+1}$. That is,

$$(4.5.1)\quad (\phi\delta\xi^{n,m,l+1},\omega)_i + \big((I\tilde{q}-q_h)^{n,m,l+1},\nabla\omega\big)_i = -\big(\phi(Ic_t^{n,m,l+1}-\partial\tilde{c}^{n,m+1}),\omega\big)_i, \qquad \omega\in\mathcal{M}_i^0,$$

where $\tilde{q}=q(\tilde{c},u)$. Also, (3.5.19) and (4.3.3) show that

$$(4.5.2)\quad \begin{aligned}\xi^{n,m,l+1}(x) &= \Lambda_i(IC^{n,m,l+1}-IC_h^{n,m,l+1})(x)\\ &= \Lambda_i(I\Xi^{n,m,l+1})(x)+\Lambda_i\big(I(C-\tilde{C})^{n,m,l+1}\big)(x) \qquad \text{for } x\in\partial\Omega_i.\end{aligned}$$

In the analysis, we will assume that l^* is a fixed integer; distinguishing the rate of asymptotic convergence with respect to δt_c, in place of Δt_c, will not be obtained below, though improvement in calculated values should be expected. Choose the test function

$$\omega=\xi^{n,m,l+1}-\big[\Lambda_i(I\Xi^{n,m,l+1})+\Lambda_i\big(I(C-\tilde{C})^{n,m,l+1}\big)\big]\in\mathcal{M}_i^0$$

in (4.5.1). After multiplying by δt_c, summing on i, n, m, and l, and canceling two terms, we obtain the equation

$$(4.5.3)\quad \begin{aligned}&\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\big\{(\phi\delta\xi^{n,m,l+1},\xi^{n,m,l+1})_m+\big((I\tilde{q}-q_h)^{n,m,l+1},\nabla\xi^{n,m,l+1}\big)_m\big\}\delta t_c\\ &\quad=-\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\big\{\big(\phi(Ic_t^{n,m,l+1}-\partial\tilde{c}^{n,m+1}),\xi^{n,m,l+1}-\Lambda_i(I(C-\tilde{C})^{n,m,l+1})\big)_i\big\}\delta t_c\\ &\qquad+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\big\{\big(\phi(\delta\xi^{n,m,l+1},\Lambda_i(I(C-\tilde{C})^{n,m,l+1})\big)_i\\ &\qquad\qquad+\big((I\tilde{q}-q_h)^{n,m,l+1},\nabla\Lambda_i(I(C-\tilde{C})^{n,m,l+1})\big)_i\big\}\delta t_c\\ &\qquad+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\big\{\big(\phi(Ic_t^{n,m,l+1}-\delta c_h^{n,m,l+1}),\Lambda_i(I\Xi^{n,m,l+1})\big)_i\\ &\qquad\qquad+\big((I\tilde{q}-q_h)^{n,m,l+1},\nabla\Lambda_i(I\Xi^{n,m,l+1})\big)_i\big\}\delta t_c.\end{aligned}$$

Let us begin our analysis of this equation by estimating the L^2_m-norm of the flux

error expression. By definitions (2.3.4) and (3.5.18), it can be written as follows:

(4.5.4)
$$\begin{aligned}(I\tilde{q}-q_h)^{n,m,l+1} &= I\big(d(u)\nabla\tilde{c}-\tilde{c}u\big)^{n,m,l+1}\\ &\quad-\big[\big(d(Eu_h^{n+\frac12})\nabla c_h^{n,m,l+1}-c_h^{n,m,l+1}Eu_h^{n+\frac12}\big)\\ &\qquad+\big(d(Eu_h^{n,m,l+1})-d(Eu_h^{n+\frac12})\big)\nabla c_h^{n,m,l}\\ &\qquad-c_h^{n,m,l}(Eu_h^{n,m,l+1}-Eu_h^{n+\frac12})\big]\\ &= d(Eu_h^{n+\frac12})\nabla\xi^{n,m,l+1}-\xi^{n,m,l+1}Eu^{n+\frac12}+\xi^{n,m,l+1}Ev^{n+\frac12}\\ &\quad+I\big(d(u)\nabla\tilde{c}\big)^{n,m,l+1}-d(Eu_h^{n+\frac12})\nabla I\tilde{c}^{n,m,l+1}\\ &\quad-\big(d(Eu_h^{n,m,l+1})-d(Eu_h^{n+\frac12})\big)\nabla c_h^{n,m,l}\\ &\quad-I(\tilde{c}u)^{n,m,l+1}+I\tilde{c}^{n,m,l+1}Eu_h^{n+\frac12}\\ &\quad+c_h^{n,m,l}(Eu_h^{n,m,l+1}-Eu_h^{n+\frac12}).\end{aligned}$$

For now we shall leave the first term on the far right-hand side above as it is. The L_m^2-norm of the second term satisfies the trivial bound

$$\|\xi^{n,m,l+1}Eu^{n+\frac12}\|_{0,m}\le Q\,\|\xi^{n,m,l+1}\|_{0,m}. \tag{4.5.5}$$

The L_m^2-norm of the third term can be estimated by using an inverse inequality which holds on the spaces $\mathcal{M}_i^L$ because of the quasi-regularity of the partition $\mathcal{T}_{h_c}(\Omega_i)$. In three space dimensions, we have that

(4.5.6)
$$\begin{aligned}\|\xi^{n,m,l+1}Ev^{n+\frac12}\|_{0,m} &\le \|Ev^{n+\frac12}\|_{0,m}\,\|\xi^{n,m,l+1}\|_{0,\infty,m}\\ &\le Q[(\|v^n\|_{0,m}+\|v^{n-1}\|_{0,m})h_c^{-\frac32}]\,\|\xi^{n,m,l+1}\|_{0,m}\end{aligned}$$

(again, if $n=0$, interpret quantities at time t_{n-1} as zero).

The fourth, fifth, and sixth terms on the far right-hand side of (4.5.4) can be combined. Since

$$I\tilde{c}^{n,m,l+1}-I\tilde{c}^{n,m,l}=\partial\tilde{c}^{n,m+1}\delta t_c,$$

we have that

(4.5.7)
$$\begin{aligned}&I\big(d(u)\nabla\tilde{c}\big)^{n,m,l+1}-d(Eu_h^{n+\frac12})\nabla I\tilde{c}^{n,m,l+1}\\ &\qquad-\big(d(Eu_h^{n,m,l+1})-d(Eu_h^{n+\frac12})\big)\nabla c_h^{n,m,l}\\ &\quad=I\big(d(u)\nabla\tilde{c}\big)^{n,m,l+1}-d(Eu_h^{n,m,l+1})\nabla I\tilde{c}^{n,m,l+1}\\ &\qquad+\big(d(Eu_h^{n,m,l+1})-d(Eu_h^{n+\frac12})\big)(\nabla\partial\tilde{c}^{n,m+1}\delta t_c+\nabla\xi^{n,m,l}).\end{aligned}$$

The L_m^2-norm of the first two terms above can be estimated easily using the Lipschitz

nature of $d(u)$ once we expand the expression as below.

(4.5.8)

$$\begin{aligned}
&\big\|I\big(d(u)\nabla\tilde{c}\big)^{n,m,l+1} - d(Eu_h^{n,m,l+1})\nabla I\tilde{c}^{n,m,l+1}\big\|_{0,m} \\
&\qquad\le \big\|I\big(d(u)\nabla\tilde{c}\big)^{n,m,l+1} - \big(d(u)\nabla\tilde{c}\big)^{n,m,l+1}\big\|_{0,m} \\
&\qquad\quad + \big\|d(u^{n,m,l+1})[\nabla\tilde{c}^{n,m,l+1} - \nabla I\tilde{c}^{n,m,l+1}]\big\|_{0,m} \\
&\qquad\quad + \big\|\big(d(u^{n,m,l+1}) - d(Eu^{n,m,l+1})\big)\nabla I\tilde{c}^{n,m,l+1}\big\|_{0,m} \\
&\qquad\quad + \big\|\big(d(Eu^{n,m,l+1}) - d(Eu_h^{n,m,l+1})\big)\nabla I\tilde{c}^{n,m,l+1}\big\|_{0,m} \\
&\qquad\le Q\big\{\big\|\big(d(u)\nabla\tilde{c}\big)_t\big\|_{L^2(J_{n,m+1};L^2_m)}(\Delta t_c)^{\frac12} + \|\nabla\tilde{c}^{n,m,l+1} - \nabla I\tilde{c}^{n,m,l+1}\|_{0,m} \\
&\qquad\quad + \|u^{n,m,l+1} - Eu^{n,m,l+1}\|_{0,m} + \|E\upsilon^{n,m,l+1}\|_{0,m}\big\} \\
&\qquad\le Q\big\{\big[\|u_t\|_{L^2(J_{n,m+1};L^2_m)} + \|\nabla\tilde{c}_t\|_{L^2(J_{n,m+1};L^2_m)}\big](\Delta t_c)^{\frac12} \\
&\qquad\quad + \|u_{tt}\|_{L^2(J_{n+1}\cap J_n;L^2_m)}(\Delta t_p)^{\frac32} + \|\upsilon^n\|_{0,m} + \|\upsilon^{n-1}\|_{0,m}\big\},
\end{aligned}$$

where, again, E is only first order accurate if $n = 0$, so $\|u_{tt}\|_{L^2(J_2\cap J_1;L^2_m)}(\Delta t_p)^{3/2}$ must be replaced by $\|u_t\|_{L^2(J_1;L^2_m)}(\Delta t_p^1)^{1/2}$ in that case.

The last term of the right-hand side of (4.5.7) is more interesting. Its L^2_m-norm can be bounded by expanding the expression and applying the inverse inequality:

(4.5.9)

$$\begin{aligned}
&\big\|\big(d(Eu_h^{n,m,l+1}) - d(Eu_h^{n+\frac12})\big)(\nabla\partial\tilde{c}^{n,m+1}\delta t_c + \nabla\xi^{n,m,l})\big\|_{0,m} \\
&\qquad= \big\|\big[\big(d(Eu_h^{n,m,l+1}) - d(Eu^{n,m,l+1})\big) + \big(d(Eu^{n+\frac12}) - d(Eu_h^{n+\frac12})\big) \\
&\qquad\quad + \big(d(Eu^{n,m,l+1}) - d(Eu^{n+\frac12})\big)\big]\big(\nabla\partial\tilde{c}^{n,m+1}\delta t_c + \nabla\xi^{n,m,l}\big)\big\|_{0,m} \\
&\qquad\le Q\big\{\big(\|E\upsilon^{n,m,l+1}\|_{0,m} + \|E\upsilon^{n+\frac12}\|_{0,m}\big)\big(\|\nabla\partial\tilde{c}^{n,m+1}\|_{0,\infty,m}\delta t_c + \|\nabla\xi^{n,m,l}\|_{0,\infty,m}\big) \\
&\qquad\quad + \|Eu^{n,m,l+1} - Eu^{n+\frac12}\|_{0,\infty,m}\big(\|\nabla\partial\tilde{c}^{n,m+1}\|_{0,m}\delta t_c + \|\nabla\xi^{n,m,l}\|_{0,m}\big)\big\} \\
&\qquad\le Q\,[(\|\upsilon^n\|_{0,m} + \|\upsilon^{n-1}\|_{0,m})h_c^{-\frac32} + \|u_t\|_{L^\infty(L^\infty_m)}\Delta t_p] \\
&\qquad\qquad \times \big(\|\nabla\tilde{c}_t\|_{L^2(J_{n,m+1};L^2_m)}(\Delta t_c)^{-\frac12}\delta t_c + \|\nabla\xi^{n,m,l}\|_{0,m}\big).
\end{aligned}$$

Note that Lemma 4.3.1 implies that

(4.5.10)

$$\begin{aligned}
\|\tilde{c}_t\|_{L^2(J_{n,m+1};H^1_m)} &\le \|c_t - \tilde{c}_t\|_{L^2(J_{n,m+1};H^1_m)} + \|c_t\|_{L^2(J_{n,m+1};H^1_m)} \\
&\le Q\,\|c\|_{H^1(J_{n,m+1};H^1_m)}.
\end{aligned}$$

Analogously, the bounds of (4.5.8)–(4.5.9) also hold for the L^2_m-norm of the last three terms on the far right-hand side of (4.5.4), provided only that the gradients of $\tilde{c}$ and c_h are replaced by the corresponding functions themselves. These bounds and

those of (4.5.5)–(4.5.6) enable us to obtain from (4.5.4) the useful estimate that

(4.5.11)
$$\begin{aligned}
&\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|(I\tilde{q}-q_h)^{n,m,l+1}-d(Eu_h^{n+\frac{1}{2}})\nabla\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\\
&\qquad\le Q\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\Big\{[1+(\|\upsilon^n\|_{0,m}^2+\|\upsilon^{n-1}\|_{0,m}^2)h_c^{-3}]\,\|\xi^{n,m,l+1}\|_{0,m}^2\\
&\qquad\quad+[\|u_t\|^2_{L^2(J_{n,m+1};L^2_m)}+\|\tilde{c}_t\|^2_{L^2(J_{n,m+1};H^1_m)}]\Delta t_c\\
&\qquad\quad+\|u_{tt}\|^2_{L^2(J_{n+1}\cap J_n;L^2_m)}(\Delta t_p)^3+\|\upsilon^n\|_{0,m}^2+\|\upsilon^{n-1}\|_{0,m}^2\\
&\qquad\quad+[(\|\upsilon^n\|_{0,m}^2+\|\upsilon^{n-1}\|_{0,m}^2)h_c^{-3}+\|u_t\|^2_{L^\infty(L^\infty_m)}(\Delta t_p)^2]\\
&\qquad\qquad\times[\|\tilde{c}_t\|^2_{L^2(J_{n,m+1};H^1_m)}(\Delta t_c)^{-1}(\delta t_c)^2+\|\xi^{n,m,l}\|^2_{1,m}]\Big\}\delta t_c\\
&\qquad\le Q\Big\{\mathcal{E}^2+\sum_{n,m}^{N,M}\big(\|\xi^n\|_{0,m}^2+\|\Xi^n\|_0^2\big)\Delta t_c\\
&\qquad\quad+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\Big\{[1+(\|\upsilon^n\|_{0,m}^2+\|\upsilon^{n-1}\|_{0,m}^2)h_c^{-3}]\|\xi^{n,m,l+1}\|_{0,m}^2\\
&\qquad\qquad+[(\|\upsilon^n\|_{0,m}^2+\|\upsilon^{n-1}\|_{0,m}^2)h_c^{-3}+\|u_t\|^2_{L^\infty(L^\infty_m)}(\Delta t_p)^2]\\
&\qquad\qquad\quad\times[\,\|c\|^2_{H^1(J_{n,m+1};H^1_m)}(\Delta t_c)^{-1}(\delta t_c)^2+\|\xi^{n,m,l}\|^2_{1,m}]\Big\}\delta t_c\Big\},
\end{aligned}$$

where we have used (4.5.10) and (4.3.6) in the second inequality above. Finally, another useful result is that

(4.5.12)
$$\begin{aligned}
&\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,m}^2\delta t_c\\
&\qquad\le Q\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\Big\{\|(I\tilde{q}-q_h)^{n,m,l+1}-d(Eu_h^{n+\frac{1}{2}})\nabla\xi^{n,m,l+1}\|_{0,m}^2\\
&\qquad\qquad+\|d(Eu^{n+\frac{1}{2}})\nabla\xi^{n,m,l+1}\|_{0,m}^2\\
&\qquad\qquad+\big\|\big(d(Eu_h^{n+\frac{1}{2}})-d(Eu^{n+\frac{1}{2}})\big)\nabla\xi^{n,m,l+1}\big\|_{0,m}^2\Big\}\delta t_c\\
&\qquad\le Q\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\Big\{\|(I\tilde{q}-q_h)^{n,m,l+1}-d(Eu_h^{n+\frac{1}{2}})\nabla\xi^{n,m,l+1}\|_{0,m}^2\\
&\qquad\qquad+[1+(\|\upsilon^n\|_{0,m}^2+\|\upsilon^{n-1}\|_{0,m}^2)h_c^{-3}]\|\nabla\xi^{n,m,l+1}\|_{0,m}^2\Big\}\delta t_c,
\end{aligned}$$

where an analogue to (4.5.6) has been used.

Now let us return to (4.5.3). The left-hand side can be bounded from below in

terms of the expression in (4.5.11). We have that

(4.5.13)
$$\begin{aligned}
&\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\big\{(\phi\delta\xi^{n,m,l+1},\xi^{n,m,l+1})_m+\big((I\tilde{q}-q_h)^{n,m,l+1},\nabla\xi^{n,m,l+1}\big)_m\big\}\delta t_c\\
&\qquad\geq\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\big\{\frac{1}{2}[(\phi\xi^{n,m,l+1},\xi^{n,m,l+1})_m-(\phi\xi^{n,m,l},\xi^{n,m,l})_m]\\
&\qquad\quad+\big(d(Eu_h^{n+\frac{1}{2}})\nabla\xi^{n,m,l+1},\nabla\xi^{n,m,l+1}\big)_m\delta t_c\\
&\qquad\quad+\big((I\tilde{q}-q_h)^{n,m,l+1}-d(Eu_h^{n+\frac{1}{2}})\nabla\xi^{n,m,l+1},\nabla\xi^{n,m,l+1}\big)_m\delta t_c\big\}\\
&\qquad\geq\frac{1}{2}\phi_*\|\xi^{N,M+1}\|_{0,m}^2+d_*\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\nabla\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\\
&\qquad\quad-Q\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|(I\tilde{q}-q_h)^{n,m,l+1}-d(Eu_h^{n+\frac{1}{2}})\nabla\xi^{n,m,l+1}\|_{0,m}^2\delta t_c.
\end{aligned}$$

Since the first two terms on the right-hand side of (4.5.3) do not depend on the approximate fracture concentration, they can be treated directly. For the first we see that

(4.5.14)
$$\begin{aligned}
&-\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\big\{\big(\phi(Ic_t^{n,m,l+1}-\partial\tilde{c}^{n,m+1}),\xi^{n,m,l+1}-\Lambda_i(I(C-\tilde{C})^{n,m,l+1})\big)_i\big\}\delta t_c\\
&\qquad\leq Q\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\big\{\|Ic_t^{n,m,l+1}-\partial\tilde{c}^{n,m+1}\|_{0,m}^2\\
&\qquad\qquad+\|\xi^{n,m,l+1}\|_{0,m}^2+\|I(C-\tilde{C})^{n,m,l+1}\|_0^2\big\}\delta t_c\\
&\qquad\leq Q\Big\{\|c_{tt}\|_{L^2(L^2_m)}^2(\Delta t_c)^2+\|c_t-\tilde{c}_t\|_{L^2(L^2_m)}^2+\|C-\tilde{C}\|_{L^\infty(L^2)}^2\\
&\qquad\qquad+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\Big\}.
\end{aligned}$$

For the second term on the right-hand side of (4.5.3), first note that summation by parts yields

(4.5.15)
$$\begin{aligned}
&\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\big(\phi(\delta\xi^{n,m,l+1},\Lambda_i(I(C-\tilde{C})^{n,m,l+1})\big)_i\delta t_c\\
&\qquad=\sum_i\big(\phi\xi^{N,M+1},\Lambda_i((C-\tilde{C})^{N,M+1})\big)_i\\
&\qquad\quad-\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\big(\phi(\xi^{n,m,l},\Lambda_i(\partial(C-\tilde{C})^{n,m+1})\big)_i\delta t_c\,;
\end{aligned}$$

hence,

(4.5.16)

$$\begin{aligned}
&\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\Big\{\big(\phi(\delta\xi^{n,m,l+1},\Lambda_i(I(C-\tilde{C})^{n,m,l+1})\big)_i\\
&\qquad\qquad+\big((I\tilde{q}-q_h)^{n,m,l+1},\nabla\Lambda_i(I(C-\tilde{C})^{n,m,l+1})\big)_i\Big\}\delta t_c\\
&\le\epsilon\Big\{\|\xi^{N,M+1}\|_{0,m}^2+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,m}^2\delta t_c\Big\}\\
&\quad+Q\Big\{\|(C-\tilde{C})^{N,M+1}\|_0^2+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\big[\,\|\xi^{n,m,l}\|_{0,m}^2+\|\partial(C-\tilde{C})^{n,m+1}\|_0^2\\
&\qquad\qquad+\|I(C-\tilde{C})^{n,m,l+1}\|_0^2\,\big]\delta t_c\Big\}\\
&\le\epsilon\Big\{\|\xi^{N,M+1}\|_{0,m}^2+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,m}\delta t_c\Big\}\\
&\quad+Q\Big\{\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\xi^{n,m,l}\|_{0,m}^2\delta t_c+\|C_t-\tilde{C}_t\|_{L^2(L^2)}^2+\|C-\tilde{C}\|_{L^\infty(L^2)}^2\Big\}.
\end{aligned}$$

Here, indeed, ϵ can be taken as small as we please. Later we will apply (4.5.12), so the Q there will need to be counteracted by a small enough ϵ here.

Finally, combining (4.5.13)–(4.5.14), (4.5.16), and Lemma 4.3.1 with (4.5.3) gives us that

(4.5.17)

$$\begin{aligned}
&\Big(\frac{1}{2}\phi_*-\epsilon\Big)\|\xi^{N,M+1}\|_{0,m}^2+d_*\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\nabla\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\\
&\qquad\le Q\Big\{\mathcal{E}^2+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\\
&\qquad\qquad+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|(I\tilde{q}-q_h)^{n,m,l+1}-d(Eu_h^{n+\frac{1}{2}})\nabla\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\Big\}\\
&\qquad+\epsilon\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,m}^2\delta t_c\\
&\qquad+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\Big\{\big(\phi(Ic_t^{n,m,l+1}-\delta c_h^{n,m,l+1}),\Lambda_i(I\Xi^{n,m,l+1})\big)_i\\
&\qquad\qquad+\big((I\tilde{q}-q_h)^{n,m,l+1},\nabla\Lambda_i(I\Xi^{n,m,l+1})\big)_i\Big\}\delta t_c,
\end{aligned}$$

where (4.5.11) bounds one of the terms on the right-hand side above. The term that is coupled to the fracture concentration error Ξ has not been estimated. We turn now to an analysis of the coupling terms.

4.6. A bound for the main matrix/fracture coupling error. The last term on the right-hand side of either (4.4.14) or (4.5.17) is too strong to bound in any direct manner. The sum of these two terms, however, can be shown to be small. We shall call this sum the main matrix/fracture coupling error. We shall now derive an expression for it. First, the last term on the right-hand side of (4.4.14) can be rewritten with the definitions (2.4.2) and (3.5.20), and then the integrals can be simplified by noting the definition of Λ_i, namely, (2.4.1). The result is

$$\begin{aligned}(4.6.1)\quad \big(g_i^{n,m+1} - g_{ih}^{n,m+1}, \Xi^{n,m+1}\big) &= -\big(\phi(c_t^{n,m+1} - \partial c_h^{n,m+1}), \Lambda_i(\Xi^{n,m+1})\big)_i \\ &\quad - \left(q^{n,m+1} - \frac{1}{l^*}\sum_{l=0}^{l^*-1} q_h^{n,m,l+1}, \nabla\Lambda_i(\Xi^{n,m+1})\right)_i .\end{aligned}$$

Now, since $I\Xi^{n,m,l+1} - \Xi^{n,m+1} = (l+1-l^*)\partial\Xi^{n,m+1}\delta t_c$, the main matrix/fracture coupling error is given by

(4.6.2)

$$\begin{aligned}&\sum_{n,m}^{N,M}\sum_i\Bigg\{\sum_{l=0}^{l^*-1}\big[\big(\phi(Ic_t^{n,m,l+1} - \delta c_h^{n,m,l+1}), \Lambda_i(I\Xi^{n,m,l+1})\big)_i \\ &\qquad + \big((I\tilde{q} - q_h)^{n,m,l+1}, \nabla\Lambda_i(I\Xi^{n,m,l+1})\big)_i\big]\delta t_c \\ &\quad - \Bigg[\big(\phi(c_t^{n,m+1} - \partial c_h^{n,m+1}), \Lambda_i(\Xi^{n,m+1})\big)_i \\ &\qquad + \left(q^{n,m+1} - \frac{1}{l^*}\sum_{l=0}^{l^*-1} q_h^{n,m,l+1}, \nabla\Lambda_i(\Xi^{n,m+1})\right)_i\Bigg]\Delta t_c\Bigg\} \\ &= \sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\big\{\big(\phi(Ic_t^{n,m,l+1} - \delta c_h^{n,m,l+1}), \Lambda_i(\partial\Xi^{n,m+1})\big)_i \\ &\qquad + \big((I\tilde{q} - q_h)^{n,m,l+1}, \nabla\Lambda_i(\partial\Xi^{n,m+1})\big)_i\big\}(l+1-l^*)(\delta t_c)^2 \\ &\quad + \sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\big\{\big(\phi(Ic_t - c_t)^{n,m+1}, \Lambda_i(\Xi^{n,m+1})\big)_i \\ &\qquad + \big(I\tilde{q}^{n,m,l+1} - q^{n,m+1}, \nabla\Lambda_i(\Xi^{n,m+1})\big)_i\big\}\delta t_c .\end{aligned}$$

If $l^* = 1$, most of the right-hand side of (4.6.2) vanishes; a term that is related to the error of the matrix concentration projection remains, but no term that contains both ξ and Ξ appears. All of the previous analyses of naturally fractured reservoir simulation [1], [2], [4] have assumed that the time step used to solve the matrix equations is the same as that used for solving the fracture equation; consequently, the main matrix/fracture coupling term could be handled relatively easily. For $l^* \geq 2$, we must give a new argument to relate quantities at matrix concentration time levels to those at the following fracture concentration time level. It should be possible to do this by completely analyzing the discrete time differences $\delta\xi^{n,m,l+1}$ and $\partial\Xi^{n,m+1}$. In the

relatively simple analysis given here, we will analyze only the matrix concentration time difference. This cannot be done in full without including an analysis of the fracture concentration time difference. Instead, as mentioned in the introduction, we will impose a mild hypothesis.

The last term on the right-hand side of (4.6.2) presents no major difficulties. Note that by definition

$$\begin{aligned} I\tilde{q}^{n,m,l+1} - q^{n,m+1} &= I\tilde{q}^{n,m,l+1} - \tilde{q}^{n,m+1} + (\tilde{q} - q)^{n,m+1} \\ &= (l + 1 - l^*)\partial\tilde{q}^{n,m+1}\delta t_c - q^{n,m+1}\big((c - \tilde{c}), u\big). \end{aligned}$$

Hence, integration by parts with (4.3.3) gives that

(4.6.3)

$$\begin{aligned} &\big(I\tilde{q}^{n,m,l+1} - q^{n,m+1}, \nabla\Lambda_i(\Xi^{n,m+1})\big)_i \\ &\quad = \big((l + 1 - l^*)\partial\tilde{q}^{n,m+1}\delta t_c, \nabla\Lambda_i(\Xi^{n,m+1})\big)_i \\ &\qquad - \big(d(u^{n,m+1})\nabla(c - \tilde{c})^{n,m+1} - (c - \tilde{c})^{n,m+1}u^{n,m+1}, \nabla\Lambda_i(\Xi^{n,m+1})\big)_i \\ &\quad = \big((l + 1 - l^*)\partial\tilde{q}^{n,m+1}\delta t_c, \nabla\Lambda_i(\Xi^{n,m+1})\big)_i \\ &\qquad + \big((c - \tilde{c})^{n,m+1}, \nabla \cdot d(u^{n,m+1})\nabla\Lambda_i(\Xi^{n,m+1})\big)_i \\ &\qquad + \big((c - \tilde{c})^{n,m+1}u^{n,m+1}, \nabla\Lambda_i(\Xi^{n,m+1})\big)_i. \end{aligned}$$

We have already observed (4.5.10), so

$$\begin{aligned} \|\partial\tilde{q}^{n,m+1}\|_{0,m} &\le \big\|\tilde{q}_t\big\|_{L^2(J_{n,m+1};L^2_m)}(\Delta t_c)^{-\frac{1}{2}} \\ &\le Q\big\{\|u_t\|_{L^2(J_{n,m+1};L^2_m)} + \|c\|_{H^1(J_{n,m+1};H^1_m)}\big\}(\Delta t_c)^{-\frac{1}{2}}, \end{aligned}$$

and the last term on the right-hand side of (4.6.2) satisfies the inequality

(4.6.4)

$$\begin{aligned} &\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\big\{\big(\phi(Ic_t^{n,m,l+1} - c_t^{n,m+1}), \Lambda_i(\Xi^{n,m+1})\big)_i \\ &\qquad\qquad + \big(I\tilde{q}^{n,m,l+1} - q^{n,m+1}, \nabla\Lambda_i(\Xi^{n,m+1})\big)_i\big\}\delta t_c \\ &\quad \le Q\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\big\{\|Ic_t^{n,m,l+1} - c_t^{n,m+1}\|^2_{0,m} + \|\Xi^{n,m+1}\|^2_0 \\ &\qquad\qquad + \|\partial\tilde{q}^{n,m+1}\|^2_{0,m}(\Delta t_c)^2 + \|(c - \tilde{c})^{n,m+1}\|^2_{0,m}\big\}\delta t_c \\ &\quad \le Q\Big\{\mathcal{E}^2 + \sum_{n,m}^{N,M}\|\Xi^{n,m+1}\|^2_0\Delta t_c\Big\}. \end{aligned}$$

We will now estimate the first term on the right-hand side of (4.6.2) by giving a partial analysis of the matrix concentration time difference. Choose the test function

$$\omega = \big\{\big[\Lambda_i(\partial\Xi^{n,m+1}) + \Lambda_i\big(\partial(C - \tilde{C})^{n,m+1}\big)\big] - \delta\xi^{n,m,l+1}\big\}(l + 1 - l^*)(\delta t_c)^2 \in \mathcal{M}_i^0$$

in (4.5.1). After combining two terms and summing on i, n, m, and l, we obtain an

expression for our term:

(4.6.5)

$$\begin{aligned}
&\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i \big\{\big(\phi(Ic_t^{n,m,l+1}-\delta c_h^{n,m,l+1}),\Lambda_i(\partial\Xi^{n,m+1})\big)_i \\
&\qquad\qquad +\big((I\tilde{q}-q_h)^{n,m,l+1},\nabla\Lambda_i(\partial\Xi^{n,m+1})\big)_i\big\}(l+1-l^*)(\delta t_c)^2 \\
&\quad=\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\big\{(\phi\delta\xi^{n,m,l+1},\delta\xi^{n,m,l+1})_m \\
&\qquad\qquad +\big((I\tilde{q}-q_h)^{n,m,l+1},\nabla\delta\xi^{n,m,l+1}\big)_m\big\}(l+1-l^*)(\delta t_c)^2 \\
&\qquad -\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\big\{\big(\phi\delta\xi^{n,m,l+1},\Lambda_i(\partial(C-\tilde{C})^{n,m+1})\big)_i \\
&\qquad\qquad -\big(\phi(Ic_t^{n,m,l+1}-\partial\tilde{c}^{n,m+1}),\delta\xi^{n,m,l+1}-\Lambda_i(\partial(C-\tilde{C})^{n,m+1})\big)_i \\
&\qquad\qquad +\big((I\tilde{q}-q_h)^{n,m,l+1},\nabla\Lambda_i(\partial(C-\tilde{C})^{n,m+1})\big)_i\big\}(l+1-l^*)(\delta t_c)^2.
\end{aligned}$$

The last term on the right-hand side above has an extra factor of δt_c, so it is easily bounded with (4.5.12). We have that

(4.6.6)

$$\begin{aligned}
&-\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i\big\{\big(\phi\delta\xi^{n,m,l+1},\Lambda_i(\partial(C-\tilde{C})^{n,m+1})\big)_i \\
&\qquad\qquad -\big(\phi(Ic_t^{n,m,l+1}-\partial\tilde{c}^{n,m+1}),\delta\xi^{n,m,l+1}-\Lambda_i(\partial(C-\tilde{C})^{n,m+1})\big)_i \\
&\qquad\qquad +\big((I\tilde{q}-q_h)^{n,m,l+1},\nabla\Lambda_i(\partial(C-\tilde{C})^{n,m+1})\big)_i\big\}(l+1-l^*)(\delta t_c)^2 \\
&\quad\le\epsilon\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\big\{\|\delta\xi^{n,m,l+1}\|_{0,m}^2+\|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,m}^2\big\}(l^*-l-1)(\delta t_c)^2 \\
&\qquad +Q\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\big\{\|\partial(C-\tilde{C})^{n,m+1})\|_0^2+\|Ic_t^{n,m,l+1}-\partial\tilde{c}^{n,m+1}\|_{0,m}^2\big\}\Delta t_c\delta t_c \\
&\quad\le\epsilon\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\big\{\|\delta\xi^{n,m,l+1}\|_{0,m}^2(l^*-l-1)(\delta t_c)^2+\|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,m}^2\delta t_c\big\} \\
&\qquad +Q\big\{\|C_t-\tilde{C}_t\|_{L^2(L^2)}^2+\|c_{tt}\|_{L^2(L_m^2)}^2(\Delta t_c)^2+\|c_t-\tilde{c}_t\|_{L^2(L_m^2)}^2\big\}\Delta t_c.
\end{aligned}$$

The first term on the right-hand side of (4.6.5) contains a nonpositive expression that helps us and an expression containing $\nabla\delta\xi^{n,m,l+1}$ that is troublesome. We would like to extract a collapsing sum from the latter expression, but the factor $(l+1-l^*)$ prevents this in general. However, it is possible to proceed by extracting a sum that partially collapses provided that $l^*\le 5$ [3] (recall that this part of the main coupling

error is zero if $l^* = 1$). We shall content ourselves here with an argument that holds for all l^*. This requires that we restrict the overall size of Δt_c as compared to h_c^2, as will be seen in the next subsection. By an inverse inequality we have that

(4.6.7)

$$\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\left((I\tilde{q}-q_h)^{n,m,l+1},\nabla\delta\xi^{n,m,l+1}\right)_m(l+1-l^*)(\delta t_c)^2$$

$$\leq \sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i \|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,i}\|\nabla\delta\xi^{n,m,l+1}\|_{0,i}(l^*-l-1)(\delta t_c)^2$$

$$\leq Q\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\sum_i \|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,i}\|\delta\xi^{n,m,l+1}\|_{0,i}h_c^{-1}(l^*-l-1)(\delta t_c)^2$$

$$\leq \sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\{Q\Delta t_c h_c^{-2}\|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,m}^2$$

$$+\epsilon\|\delta\xi^{n,m,l+1}\|_{0,m}^2(l^*-l-1)\delta t_c\}\delta t_c.$$

Finally, we can combine (4.6.4)–(4.6.7) with Lemma 4.3.1 to obtain that the main coupling error (4.6.2) is bounded by the expression

(4.6.8)

$$(\epsilon-\phi_*)\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\delta\xi^{n,m,l+1}\|_{0,m}^2(l^*-l-1)(\delta t_c)^2+Q\left\{\mathcal{E}^2+\sum_{n,m}^{N,M}\|\Xi^{n,m+1}\|_0^2\Delta t_c\right\}$$

$$+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}(\epsilon+Q\Delta t_c h_c^{-2})\|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,m}^2\delta t_c.$$

4.7. The combined analysis of the concentration error equations. We are now ready to complete the analysis of the concentration equations by applying an induction argument to the bounds derived in the previous three subsections. Add (4.4.14) to (4.5.17) and bound the main coupling error by (4.6.8). We obtain the single

inequality

(4.7.1)
$$\begin{aligned}
&\frac{1}{2}\Phi_*\|\Xi^{N,M+1}\|_0^2+\left(\frac{1}{2}\phi_*-\epsilon\right)\|\xi^{N,M+1}\|_{0,m}^2\\
&\qquad+(D_*-\epsilon)\sum_{n,m}^{N,M}\|\nabla\Xi^{n,m+1}\|_0^2\Delta t_c+d_*\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\nabla\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\\
&\qquad+(\phi_*-\epsilon)\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\delta\xi^{n,m,l+1}\|_{0,m}^2(l^*-l-1)(\delta t_c)^2\\
&\quad\le Q\Bigg\{\mathcal{E}^2+\sum_{n,m}^{N,M}\|\Xi^{n,m+1}\|_0^2\Delta t_c+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\\
&\qquad\qquad+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|(I\tilde{q}-q_h)^{n,m,l+1}-d(Eu_h^{n+\frac{1}{2}})\nabla\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\Bigg\}\\
&\qquad+\epsilon\sum_{n,m}^{N,M}(\|\Upsilon^n\|_0^2+\|\Upsilon^{n-1}\|_0^2)H_c^{-3}\|\nabla\Xi^{n,m}\|_0^2\Delta t_c\\
&\qquad+(\epsilon+Q\Delta t_c h_c^{-2})\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,m}^2\delta t_c.
\end{aligned}$$

We need to control the right-hand side. Assume the asymptotic relations

$$\Delta t_c h_c^{-2}=o(1),\qquad \mathcal{E}^2h_c^{-3}=o(1),\qquad \mathcal{E}^2H_c^{-3}=o(1),\tag{4.7.2}$$

as the discretization parameters tend to zero. Also make the induction hypotheses that

$$(\|v^n\|_{0,m}^2+\|v^{n-1}\|_{0,m}^2)h_c^{-3}=o(1),\qquad(\|\Upsilon^n\|_0^2\ +\ \|\Upsilon^{n-1}\|_0^2)H_c^{-3}=o(1).\tag{4.7.3}$$

From (4.5.11) and (4.5.12), these assumptions are enough to give us the bounds

$$\begin{aligned}
&\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|(I\tilde{q}-q_h)^{n,m,l+1}-d(Eu_h^{n+\frac{1}{2}})\nabla\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\\
&\qquad\le Q_1\Bigg\{\mathcal{E}^2+\sum_{n,m}^{N,M}\|\Xi^{n,m+1}\|_0^2\Delta t_c+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\\
&\qquad\qquad+o(1)\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\nabla\xi^{n,m,l}\|_{0,m}^2\delta t_c\Bigg\}
\end{aligned}\tag{4.7.4}$$

and

$$\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|(I\tilde{q}-q_h)^{n,m,l+1}\|_{0,m}^2\delta t_c \tag{4.7.5}$$
$$\leq Q_2\Big\{\mathcal{E}^2+\sum_{n,m}^{N,M}\|\Xi^{n,m+1}\|_0^2\Delta t_c+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\xi^{n,m,l+1}\|_{0,m}^2\delta t_c + \sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\nabla\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\Big\}.$$

By induction, then, we can assert that (4.7.1) implies that

$$\begin{aligned}&\|\Xi^{N,M+1}\|_0^2+\|\xi^{N,M+1}\|_{0,m}^2\\ &\qquad+\sum_{n,m}^{N,M}\|\nabla\Xi^{n,m+1}\|_0^2\Delta t_c+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\nabla\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\\ &\leq Q_3\Big\{\mathcal{E}^2+\sum_{n,m}^{N,M}\|\Xi^{n,m+1}\|_0^2\Delta t_c+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\Big\},\end{aligned} \tag{4.7.6}$$

provided only that the induction parameters are small enough. Gronwall's inequality can now be applied to see that in fact

$$\begin{aligned}&\|\Xi^{N,M+1}\|_0^2+\|\xi^{N,M+1}\|_{0,m}^2\\ &\qquad+\sum_{n,m}^{N,M}\|\nabla\Xi^{n,m+1}\|_0^2\Delta t_c+\sum_{n,m}^{N,M}\sum_{l=0}^{l^*-1}\|\nabla\xi^{n,m,l+1}\|_{0,m}^2\delta t_c\\ &\leq Q_4\mathcal{E}^2.\end{aligned} \tag{4.7.7}$$

It remains only to verify the induction hypotheses. But it is clear that (4.3.5)–(4.3.6) and the initialization requirements (3.5.14)–(3.5.15) enable us to start the induction, and (4.3.5)–(4.3.6) with (4.7.7) at no more than two previous pressure time levels enable us to continue the induction.

Acknowledgments

This work contains most of the results of the author's doctoral dissertation at the University of Chicago [3] under the direction of Jim Douglas, Jr. The author would like to thank and acknowledge him for many helpful conversations pertaining to this work. The National Science Foundation partially supported this work, and it was completed while the author was visiting the Institute for Mathematics and its Applications at the University of Minnesota.

REFERENCES

[1] T. Arbogast, *Analysis of the simulation of single phase flow through a naturally fractured reservoir*, SIAM J. Numer. Anal., 26 (1989), to appear.

[2] T. Arbogast, *The double porosity model for single phase flow in naturally fractured reservoirs*, in Numerical Simulation in Oil Recovery, M. F. Wheeler, ed., The IMA Volumes in Mathematics and its Applications 11, Springer-Verlag, Berlin and New York, 1988, pp. 23–45.

[3] T. Arbogast, *Ph.D. Thesis*, The University of Chicago, Chicago, Illinois, 1987.

[4] T. Arbogast, J. Douglas, Jr., and J. E. Santos, *Two-phase immiscible flow in naturally fractured reservoirs*, in Numerical Simulation in Oil Recovery, M. F. Wheeler, ed., The IMA Volumes in Mathematics and its Applications 11, Springer-Verlag, Berlin and New York, 1988, pp. 47–66.

[5] G. I. Barenblatt, Iu. P. Zheltov, and I. N. Kochina, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks [strata]*, Prikl. Mat. Mekh., 24 (1960), pp. 852–864; J. Appl. Math. Mech., 24 (1960), pp. 1286–1303.

[6] F. Brezzi, J. Douglas, Jr., R. Durán, and M. Fortin, *Mixed finite elements for second order elliptic problems in three variables*, Numer. Math., 51 (1987), pp. 237–250.

[7] F. Brezzi, J. Douglas, Jr., M. Fortin, and L. D. Marini, *Efficient rectangular mixed finite elements in two and three space variables*, R.A.I.R.O. Modél. Math. Anal. Numér., 21 (1987), pp. 581–604.

[8] J. Douglas, Jr., *Numerical methods for the flow of miscible fluids in porous media*, in Numerical Methods in Coupled Systems, R. W. Lewis, P. Bettess, and E. Hinton, eds., John Wiley and Sons Ltd, London, 1984, pp. 405–439.

[9] J. Douglas, Jr., R. E. Ewing, and M. F. Wheeler, *The approximation of the pressure by a mixed method in the simulation of miscible displacement*, R.A.I.R.O. Anal. Numér., 17 (1983), pp. 17–33.

[10] J. Douglas, Jr., R. E. Ewing, and M. F. Wheeler, *A time-discretization procedure for a mixed finite element approximation of miscible displacement in porous media*, R.A.I.R.O. Anal. Numér., 17 (1983), pp. 249–265.

[11] J. Douglas, Jr., P. J. Paes Leme, T. Arbogast, and T. Schmitt, *Simulation of flow in naturally fractured reservoirs*, Paper SPE 16019, in Proceedings, Ninth SPE Symposium on Reservoir Simulation, Society of Petroleum Engineers, Dallas, Texas, 1987, pp. 271–279.

[12] J. Douglas, Jr., and J. E. Roberts, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp., 44 (1985), pp. 39–52.

[13] J. Douglas, Jr., and T. F. Russell, *Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures*, SIAM J. Numer. Anal., 19 (1982), pp. 871–885.

[14] J. Douglas, Jr., M. F. Wheeler, B. L. Darlow, and R. P. Kendall, *Self-adaptive finite element simulation of miscible displacement in porous media*, Comp. Meth. Appl. Mech. Eng., 47 (1984), pp. 131–159.

[15] J. Douglas, Jr., and Y. Yuan, *Numerical simulation of immiscible flow in porous media based on combining the method of characteristics with mixed finite element procedures*, in Numerical Simulation in Oil Recovery, M. F. Wheeler, ed., The IMA Volumes in Mathematics and its Applications 11, Springer-Verlag, Berlin and New York, 1988, pp. 119–131.

[16] R. Durán, *On the approximation of miscible displacement in porous media by a method of characteristics combined with a mixed method*, SIAM J. Numer. Anal, 25 (1988), pp. 989–1001.

[17] R. E. Ewing, T. F. Russell, and M. F. Wheeler, *Simulation of miscible displacement using mixed methods and a modified method of characteristics*, Paper SPE 12241, in Proceedings, Seventh SPE Symposium on Reservoir Simulation, Society of Petroleum Engineers, Dallas, Texas, 1983, pp. 71–81.

[18] R. E. Ewing, T. F. Russell, and M. F. Wheeler, *Convergence analysis of an approximation of miscible displacement in porous media by mixed finite elements and a modified method of characteristics*, Comp. Meth. Appl. Mech. Eng., 47 (1984), pp. 73–92.

[19] R. E. Ewing and M. F. Wheeler, *Galerkin methods for miscible displacement problems in porous media*, SIAM J. Numer. Anal., 17 (1980), pp. 351–365.

[20] R. E. Ewing and M. F. Wheeler, *Galerkin methods for miscible displacement problems with point sources and sinks–unit mobility ratio case*, in Mathematical Methods in Energy Research, K. I. Gross, ed., Society for Industrial and Applied Mathematics, Philadelphia, 1984, pp. 40–58.

[21] H. Kazemi, *Pressure transient analysis of naturally fractured reservoirs with uniform fracture distribution*, Soc. Pet. Eng. J. (1969), pp. 451–462.

[22] J. C. Nedelec, *Mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 35 (1980), pp. 315–341.

[23] D. W. Peaceman, *Improved treatment of dispersion in numerical calculation of multidimensional miscible displacement*, Soc. Pet. Eng. J. (1966), pp. 213–216.

[24] P. A. Raviart and J. M. Thomas, *A mixed finite element method for 2nd order elliptic problems*, in Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics 606, Springer-Verlag, Berlin and New York, 1977, pp. 292–315.

[25] T. F. Russell, *Time stepping along characteristics with incomplete iteration for a Galerkin approximation of miscible displacement in porous media*, SIAM J. Numer. Anal., 22 (1985), pp. 970–1013.

[26] P. H. Sammon, *Numerical approximations for a miscible displacement process in porous media*, SIAM J. Numer. Anal., 23 (1986), pp. 508–542.

[27] F. Sonier, P. Souillard, and F. T. Blaskovich, *Numerical simulation of naturally fractured reservoirs*, Paper SPE 15627, in Proceedings, 61st Annual Technical Conference and Exhibition of the Society of Petroleum Engineers, Society of Petroleum Engineers, Dallas, Texas, 1986.

[28] A. de Swaan O., *Analytic solutions for determining naturally fractured reservoir properties by well testing*, Soc. Pet. Eng. J. (1976), pp. 117–122.

[29] J. E. Warren and P. J. Root, *The behavior of naturally fractured reservoirs*, Soc. Pet. Eng. J. (1963), pp. 245–255.

[30] M. F. Wheeler, *A priori* L_2 *error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal., 10 (1973), pp. 723–759.